3/2 Colimits

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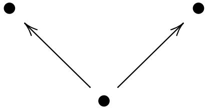
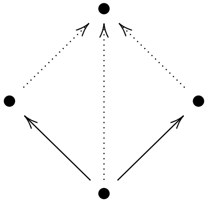
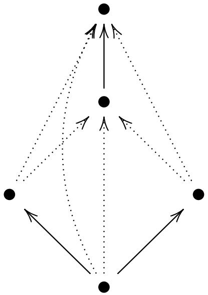
A *diagram* is a set of objects and a set of arrows between them.



A *cocone* for a diagram puts the objects in it together while taking into account how they are related in the diagram.



A colimit is a minimal cocone (unique up to isomorphism).



Colimits in Set

Theorem

Set *is co-complete.*

Colimit of an arbitrary diagram in **Set**:

- disjoint union of all sets in the diagram
- factored by the equivalence generated by the pairs $(x, f_i(x))$, with i an arrow in the diagram
- structural morphisms are factorizations

Colimits in Set: example

$$\begin{tikzcd} \{a\} \arrow[r] \arrow[d] & \{a, b\} \\ \{a, b\} & \end{tikzcd}$$

Colimits in Set: example

$$
\begin{tikzcd}
\{a\} \arrow[r] \arrow[d] & \{a, b\} \arrow[d] \\
\{a, b\} \arrow[r] & \{(1, a), (2, a), (3, a), (2, b), (3, b)\} \arrow[d] \\
& \{\langle (1, a), (2, a), (3, a)\rangle, \langle (2, b)\rangle, \langle (3, b)\rangle\}
\end{tikzcd}
$$

$$\cong$$

$$\{a, b_1, b_2\}$$

$$\cong$$

$$\{a, 2 : b, 3 : b\}$$

$$\cong$$

$$\{a, gn\_b2, gn\_b3\}$$

Fact: the category of sets and partial functions is isomorphic to the category of pointed sets:

$$X \xrightarrow{h_\perp} Y$$

Colimits in Pfn

Fact: the category of sets and partial functions is isomorphic to the category of pointed sets:

$$\begin{tikzcd}
X \arrow[rr, "h"] & & Y \\
& \{\perp\} \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Colimits in Pfn: colimits in the category of pointed sets

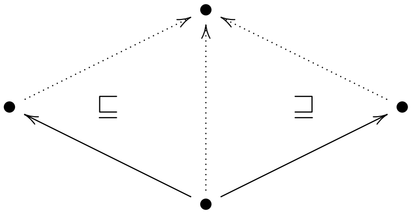
Example in Hets

- D2.1, footnote 14

Assumption: all diagrams are strict.

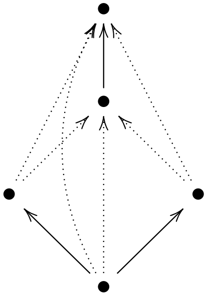
Definition

Given $D : \mathcal{J} \to (C, \sqsubseteq)$, a lax cocone is $(O, \{c_X : D(X) \to O\}_{X \in |\mathcal{J}|})$ such that $D(f); c_Y \sqsubseteq c_X$ for each $f : X \to Y \in \mathcal{J}$



Definition

Given $D : \mathcal{J} \to (C, \sqsubseteq)$, a 3/2-colimit is a lax cocone $(O, \{c_X : D(X) \to O\}_{X \in |\mathcal{J}|})$ for D such that for every lax cocone $(O', \{d_X : D(X) \to O'\}_{X \in |\mathcal{J}|})$ for D, the set of all $\lambda : O \to O'$ such that $c_X ; \lambda \sqsubseteq d_X$ has a maximum element on $\sqsubseteq$.



Theorem (D2.1)

Given $D : \mathcal{J} \to (Pfn, \sqsubseteq)$ and a lax cocone $c = (O, \{c_X : D(X) \to O\}_{X \in |\mathcal{J}|})$ for D.
c is 3/2-colimit iff c is jointly epimorphic, i.e. if for each node X $c_X ; h_1 = c_X ; h_2$, then $h_1 = h_2$.

Theorem (D2.1)

Given $D : \mathcal{J} \to (Pfn, \sqsubseteq)$ *and a lax cocone* $c = (O, \{c_X : D(X) \to O\}_{X \in |\mathcal{J}|})$ *for* D.
c *is 3/2-colimit iff* c *is jointly epimorphic, i.e. if for each node* X $c_X ; h_1 = c_X ; h_2$, *then* $h_1 = h_2$.

Corollary

Every ordinary colimit in Pfn is a 3/2 colimit.

3/2 colimits in Pfn

Theorem (D2.1)

Given $D : \mathcal{J} \to (Pfn, \sqsubseteq)$ and a lax cocone $c = (O, \{c_X : D(X) \to O\}_{X \in |\mathcal{J}|})$ for D.
c is 3/2-colimit iff c is jointly epimorphic, i.e. if for each node X $c_X ; h_1 = c_X ; h_2$, then $h_1 = h_2$.

Corollary

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3/2 colimits are stable under post-composition with epis.

Examples

An ordinary colimit is a 3/2 colimit:

$$\begin{tikzcd}
 & \{y, s\} & \\
\{x, y\} \arrow[ur, dotted, "y \mapsto y"] & & \{s, t\} \arrow[ul, dotted, "s \mapsto s"'] \\
 & \{a, b\} \arrow[ul, "_ \mapsto x"] \arrow[ur, "b \mapsto t"'] \arrow[uu, dotted] &
\end{tikzcd}$$

Examples

This is another 3/2 colimit for the same diagram:

$$\begin{tikzcd}
& \{y, s, \alpha\} & \\
\{x, y\} \arrow[ur, dotted, "y \mapsto y, x \mapsto \alpha"] & & \{s, t\} \arrow[ul, dotted, "s \mapsto s, t \mapsto \alpha"'] \\
& \{a, b\} \arrow[ul, "_ \mapsto x"] \arrow[ur, "b \mapsto t"'] \arrow[uu, dotted, "_ \mapsto \alpha"'] &
\end{tikzcd}$$

Examples

This is not a 3/2-colimit because it's not jointly epi:

$$\begin{tikzcd}
 & \{y, s, z\} & \\
\{x, y\} \arrow[ur, dotted, "y\mapsto y"] & & \{s, t\} \arrow[ul, dotted, "s\mapsto s"'] \\
 & \{a, b\} \arrow[ul, "_\mapsto x"] \arrow[ur, "b\mapsto t"'] \arrow[uu, dotted] &
\end{tikzcd}$$

Examples

This is not even a lax cocone:

$$
\begin{tikzcd}
 & \{y, s, b\} & \\
\{x, y\} \arrow[ur, dotted, "y \mapsto y"] & & \{s, t\} \arrow[ul, dotted, "s \mapsto s, t \mapsto b"'] \\
 & \{a, b\} \arrow[ul, "\_ \mapsto x"'] \arrow[ur, "b \mapsto t"] \arrow[uu, dotted] &
\end{tikzcd}
$$

Ordering on 3/2 colimits

Definition

Given $D : \mathcal{J} \to (Pfn, \sqsubseteq)$ and two 3/2 colimits $c_1 = (O_1, \{c^1_X : D(X) \to O_1\}_{X \in |\mathcal{J}|})$ and $c_2 = (O_2, \{c^2_X : D(X) \to O_2\}_{X \in |\mathcal{J}|})$ for D, $c_1 \sqsubseteq c_2$ if there exists $e : O_2 \to O_1$ such that e is epi and $c_2 ; e = c_1$.

$$
\begin{tikzcd}
 & \{y, s\} & \\
 & \{y, s, \alpha\} \arrow[u] & \\
\{x, y\} \arrow[ur, dotted, "y \mapsto y, x \mapsto \alpha"] & & \{s, t\} \arrow[ul, dotted, "s \mapsto s, t \mapsto \alpha"'] \\
 & \{a, b\} \arrow[ul, "\_ \mapsto x"] \arrow[ur, "b \mapsto t"'] \arrow[uu, dotted, "\_ \mapsto \alpha"'] &
\end{tikzcd}
$$

Constructing maximum 3/2 colimits

Rule 1: for every $D(f) : D(x) \mapsto D(y)$, add to $D(y)$ the elements of $D(x)$ mapped by $D(f)$ to $\perp$ and modify $D(f)$ to be the identity on these elements.

$$
\begin{tikzcd}
& \{y, s, \alpha\} & \\
\{x, y\} \arrow[ur, dotted, "y \mapsto y, x \mapsto \alpha"] & & \{s, t, a\} \arrow[ul, dotted, "s \mapsto s, \_ \mapsto \alpha"'] \\
& \{a, b\} \arrow[ul, "\_ \mapsto x"] \arrow[ur, "b \mapsto t, a \mapsto a"'] \arrow[uu, dotted, "\_ \mapsto \alpha"'] &
\end{tikzcd}
$$

Constructing maximum 3/2 colimits

Rule 2: for every $D(x_1) \stackrel{D(f_1)}{\rightarrow} D(y) \stackrel{D(f_2)}{\leftarrow} D(x_2)$, remove from $D(y)$ the images of $D(f_1)$ and $D(f_2)$ and add to it distinct copies of them (generalizes to the n-ary case):

$$
\begin{tikzcd}
& \{a\} & & \{a\} & \\
\{a\} \arrow[ur] & & \{a\} \arrow[ul] \arrow[ur] & & \{a\} \arrow[ul]
\end{tikzcd}
$$

Constructing maximum 3/2 colimits

Rule 2: for every $D(x_1) \overset{D(f_1)}{\rightarrow} D(y) \overset{D(f_2)}{\leftarrow} D(x_2)$, remove from $D(y)$ the images of $D(f_1)$ and $D(f_2)$ and add to it distinct copies of them (generalizes to the n-ary case):

$$
\begin{tikzcd}
 & & & \{a_1, a_2 = a_3, a_4\} & & & \\
 & \{a_1, a_2\} \arrow[urr, dotted] & & & & \{a_3, a_4\} \arrow[ull, dotted] & \\
\{a\} \arrow[ur] & & & \{a\} \arrow[ull] \arrow[urr] \arrow[uu, dotted] & & & \{a\} \arrow[ul]
\end{tikzcd}
$$

Generating 3/2 colimits

- construct the maximum 3/2 colimit c
- by composing it with an epi e, we get a 3/2 colimit
- e can identify elements
- e can 'delete' elements by mapping them to $\perp$

CASL with partial signature morphisms

- signatures: sorts, partial functions, predicates. No total functions!
- morphisms: partial maps taking sorts to sorts, functions to functions, predicates to predicates
- sentence translation:
  - replace missing symbols with undefined or false
  - $x \leq y \vee x + y = 3 + z$ is mapped along a morphism that undefines $\leq$ and $+$ to $\mathit{false} \vee x + y = \perp$
- model reduction:
  - use empty carrier/predicate/function for undefined symbols
  - the reduct of a model M along a morphism that undefines $\leq$ and $+$ inteprets $\leq$ as the predicate that never holds and $+$ as the partial function that is defined nowhere