Logik für Informatiker
Logic for computer scientists

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WiSe 2013/14

Proof methods for quantifiers

Proof methods for quantifiers

Universal elimination

Universal statments can be instantiated to any object.

From $\forall x S(x)$, we may infer $S(c)$.

Existential introduction

If we have established a statement for an instance, we can also establish the corresponding existential statement.

From $S(c)$, we may infer $\exists x S(x)$.

Example

$\forall x[\mathrm{Cube}(x) \to \mathrm{Large}(x)]$

$\forall x[\mathrm{Large}(x) \to \mathrm{LeftOf}(x, b)]$

$\mathrm{Cube}(d)$

$\exists x[\mathrm{Large}(x) \land \mathrm{LeftOf}(x, b)]$

Existential elimination

From $\exists x S(x)$, we can infer things by assuming $S(c)$ in a subproof, if c is a new name not used otherwise.
Example: Scotland Yard searched a serial killer. The did not know who he was, but for their reasoning, they called him "Jack the ripper".
This would have been an unfair procedure if there had been a real person named Jack the ripper.

Example

$$\forall x[\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)]$$
$$\forall x[\mathrm{Large}(x) \rightarrow \mathrm{LeftOf}(x, b)]$$
$$\exists x \mathrm{Cube}(x)$$

$$\exists x[\mathrm{Large}(x) \land \mathrm{LeftOf}(x, b)]$$

Universal generalization (introduction)

If we introduce a new name c that is not used elsewhere, and can prove $S(c)$, then we can also infer $\forall x S(x)$.

Example:

Theorem Every positive even number is the sum of two odd numbers.

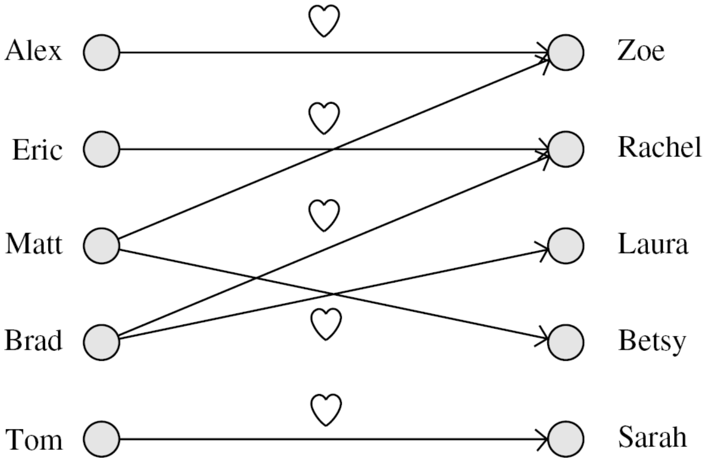
Proof Let $n > 0$ be even, i.e. $n = 2m$ with $m > 0$. If m is odd, then $m + m = n$ does the job. If m is even, consider $(m - 1) + (m + 1) = n$.

Arguments involving multiple quantifiers

$$\begin{array}{|l} \exists y[\mathrm{Girl}(y) \wedge \forall x(\mathrm{Boy}(x) \rightarrow \mathrm{Likes}(x,y))] \\ \hline \forall x[\mathrm{Boy}(x) \rightarrow \exists y(\mathrm{Girl}(y) \wedge \mathrm{Likes}(x,y))] \end{array}$$

$$\begin{array}{|l} \forall x[\mathrm{Boy}(x) \rightarrow \exists y(\mathrm{Girl}(y) \wedge \mathrm{Likes}(x,y))] \\ \hline \exists y[\mathrm{Girl}(y) \wedge \forall x(\mathrm{Boy}(x) \rightarrow \mathrm{Likes}(x,y))] \end{array}$$

A (counter)example



Universal Elimination (∀ Elim)

$$
\begin{array}{r|l}
 & \forall x\, S(x) \\
 & \vdots \\
\triangleright & S(c)
\end{array}
$$

Existential Introduction ($\exists$ Intro)

$$
\begin{array}{l|l}
 & S(c) \\
 & \vdots \\
\triangleright & \exists x\, S(x)
\end{array}
$$

Example: $\forall$-**Elim** and $\exists$-**Intro**

$$
\begin{array}{|l}
\forall x[\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)] \\
\forall x[\mathrm{Large}(x) \rightarrow \mathrm{LeftOf}(x, b)] \\
\mathrm{Cube}(d) \\
\hline
\exists x[\mathrm{Large}(x) \land \mathrm{LeftOf}(x, b)]
\end{array}
$$

Existential Elimination (∃ Elim):

$$
\begin{array}{l|l}
 & \exists x\, S(x) \\
 & \vdots \\
 & \quad \boxed{c}\ S(c) \\
 & \quad \vdots \\
 & \quad Q \\
\triangleright & Q
\end{array}
$$

Where c does not occur outside the subproof where it is introduced.

Example: ∃-**Elim**

$$
\begin{array}{|l}
\forall x[\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)] \\
\forall x[\mathrm{Large}(x) \rightarrow \mathrm{LeftOf}(x, b)] \\
\exists x\ \mathrm{Cube}(x) \\
\hline
\exists x[\mathrm{Large}(x) \land \mathrm{LeftOf}(x, b)]
\end{array}
$$

General Conditional Proof ($\forall$ Intro):

$\boxed{c}$ $P(c)$	
$\vdots$	
$Q(c)$	
$\triangleright$ $\forall x\,(P(x) \to Q(x))$	

Where c does not occur outside the subproof where it is introduced.

Example: General Conditional Proof

$$\begin{array}{|l} \forall x[\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)] \\ \forall x[\mathrm{Large}(x) \rightarrow \mathrm{LeftOf}(x, b)] \\ \hline \forall x[\mathrm{Cube}(x) \rightarrow \mathrm{LeftOf}(x, b)) \end{array}$$

Universal Introduction (∀ Intro):

$$\begin{array}{l|l} & \begin{array}{|l} \boxed{c} \\ \hline \vdots \\ P(c) \end{array} \\ \triangleright & \forall x\, P(x) \end{array}$$

Where c does not occur outside the subproof where it is introduced.

Prenex normal form (reminder)

$$\exists x \mathrm{Cube}(x) \rightarrow \forall y \mathrm{Small}(y)$$

$$\forall x \forall y (\mathrm{Cube}(x) \rightarrow \mathrm{Small}(y))$$

Example with multiple quantifiers

$$\begin{array}{|l} \exists y[\mathrm{Girl}(y) \land \forall x(\mathrm{Boy}(x) \rightarrow \mathrm{Likes}(x, y))] \\ \hline \forall x[\mathrm{Boy}(x) \rightarrow \exists y(\mathrm{Girl}(y) \land \mathrm{Likes}(x, y))] \end{array}$$

Example: de Morgan's Law

$$\begin{array}{|l} \neg\forall x\ P(x) \\ \hline \exists x\ \neg P(x) \end{array}$$

(is not valid in intuitionistic logic, only in classical logic)

Example: The Barber Paradox

$$\exists z\ \exists x\ [ManOf(x,z) \land \forall y\ (ManOf(y,z) \rightarrow (Shave(x,y) \leftrightarrow \neg Shave(y,y)))]$$

$$\perp$$

XMas

Existence of Santa Clause

Theorem. Santa Clause exists.

Proof.

Assume to the contrary, that Santa Clause does not exist.
By $\exists$-Intro, there exists something that does not exist.
This is a contradiction. Hence, the assumption that Santa Clause does not exist must be wrong.
Thus, Santa Clause exists. $\square$

All reindeers have the same color

Theorem. Any number of reindeers have the same color.

Proof. By induction.

Basis: one reindeer has the same color (obviously!).

Inductive step: suppose that any collection of n reindeers has the same color. We need to show that $n+1$ reindeers have the same color, too. By induction hypothesis, the first n reindeers have the same color. Take out the last reindeer of these and replace it with the $n+1$st. Again by induction hypothesis, these have the same color. Hence, all $n+1$ reindeers have the same color. □

Why the date of XMas cannot be surprising

Son: It is boring that XMas always is on the 24th.
Father: OK. This year, we will celebrate XMas on a day in the week from 23th to 29th. You will not know the date beforehand.
Son: Good! Then it cannot be the 29th — if we hadn't celebrated it until the 28th, I would know beforehand that it must be the 29th, since this is the last day of the week!
Moreover, it cannot be the 28th — if we hadn't celebrated it until the 27th, I would know beforehand that it must be the 28th (the 29th already has been excluded above).
Son (cont'd): Similarly, it can be neither the 27th, nor the 26th, nor the 25th, nor the 24th, nor the 23th.
Hence, you cannot fulfill you promise that I won't know the date beforehand.
Father: You will see, you won't know the date beforehand.

Why the date of XMas can be surprising

After all, XMas was celebrated on the 27th.
The son was quite surprised.

A scheduling problem

A camel must travel 1000 miles across a desert to the nearest city. She has 3000 bananas but can only carry 1000 at a time. For every mile she walks, she needs to eat a banana. What is the maximum number of bananas she can transport to the city?