Logik für Informatiker
Logic for computer scientists

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Multiple quantifiers

Multiple quantifiers in Aristotelian forms

Some cube is left of a tetrahedron

$$\exists x \exists y [Cube(x) \wedge Tet(y) \wedge LeftOf(x, y)]$$

$$\exists x [Cube(x) \wedge \exists y (Tet(y) \wedge LeftOf(x, y))]$$

Every cube is left of every tetrahedron

$$\forall x \forall y [(Cube(x) \wedge Tet(y)) \rightarrow LeftOf(x, y)]$$

$$\forall x [Cube(x) \rightarrow \forall y (Tet(y) \rightarrow LeftOf(x, y))]$$

Multiple quantifiers and conversational implicature

What is the meaning of

$$\forall x \forall y[(Cube(x) \land Cube(y)) \rightarrow (LeftOf(x,y) \lor RightOf(x,y))] \quad ?$$

What is the meaning of

$$\exists x \exists y(Cube(x) \land Cube(y)) \quad ?$$

Mixed quantifiers

Every cube is to the left of a tetrahedron.

$$\forall x[Cube(x) \rightarrow \exists y(Tet(y) \land LeftOf(x, y))]$$

$$\forall x \exists y[Cube(x) \rightarrow (Tet(y) \land LeftOf(x, y))]$$

Order of mixed quantifiers

$$\forall x \exists y \ Likes(x, y)$$

is very different from

$$\exists y \forall x \ Likes(x, y)$$

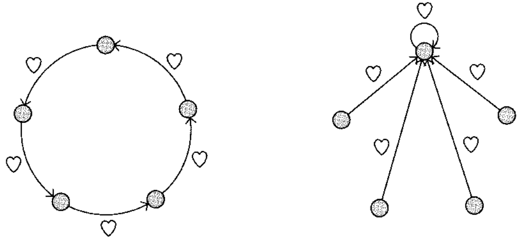

Figure 11.1: A circumstance in which $\forall x \exists y$ Likes(x, y) holds versus one in which $\exists y \forall x$ Likes(x, y) holds. It makes a big difference to someone!

There is exactly one . . .

There is exactly one cube.

$$\exists x(Cube(x) \land \forall y(Cube(y) \rightarrow y = x))$$

Step-by-step translation

Each cube is to the left of a tetrahedron.

$$\rightsquigarrow$$

$$\forall x(Cube(x) \rightarrow x \text{ is-to-the-left-of-a-tetrahedron})$$

$$x \text{ is-to-the-left-of-a-tetrahedron} \rightsquigarrow \exists y(Tet(y) \land LeftOf(x, y))$$

$$\forall x(Cube(x) \rightarrow x \text{ is-to-the-left-of-a-tetrahedron})$$

$$\rightsquigarrow$$

$$\forall x(Cube(x) \rightarrow \exists y(Tet(y) \land LeftOf(x, y)))$$

Paraphrasing can be necessary

Every farmer who owns a donkey beats it.

$\forall x(Farmer(x) \wedge \exists y(Donkey(y) \wedge Owns(x, y)) \rightarrow Beats(x, y))$ wrong!

Paraphrase:
Every donkey owned by any farmer is beaten by them.

$$\forall x(Donkey(x) \rightarrow \forall y((Farmer(y) \wedge Owns(y, x)) \rightarrow Beats(y, x)))$$

Ambiguity and context sensitivity

Every minute a man is mugged in New York City.
We are going to interview him tonight.

weak reading:

$$\forall x(Minute(x) \rightarrow \exists y(Man(y) \wedge MuggedDuring(y, x)))$$

strong reading:

$$\exists y(Man(y) \wedge \forall x(Minute(x) \rightarrow MuggedDuring(y, x)))$$

Prenex Form

Every cube to the left of a tetrahedron is in back of a dodecahedron

$$\forall x[(Cube(x) \wedge \exists y(Tet(y) \wedge LeftOf(x,y))) \to \exists y(Dodec(y) \wedge BackOf(x,y))]$$

Conversion to prenex from shifts all quantifiers to the top-level:

$$\forall x \forall y \exists z[(Cube(x) \wedge Tet(y) \wedge LeftOf(x,y)) \to (Dodec(z) \wedge BackOf(x,z))]$$

Prenex Form: Rules for conjunctions and disjunctions

$$\forall x Q \wedge P \rightsquigarrow \forall x (Q \wedge P) \qquad \exists x Q \wedge P \rightsquigarrow \exists x (Q \wedge P)$$

$$P \wedge \forall x Q \rightsquigarrow \forall x (P \wedge Q) \qquad P \wedge \exists x Q \rightsquigarrow \exists x (P \wedge Q)$$

$$\forall x Q \vee P \rightsquigarrow \forall x (Q \vee P) \qquad \exists x Q \vee P \rightsquigarrow \exists x (Q \vee P)$$

$$P \vee \forall x Q \rightsquigarrow \forall x (P \vee Q) \qquad P \vee \exists x Q \rightsquigarrow \exists x (P \vee Q)$$

Note that x must not be a free variable in P.
If x is a free variable in P, we can achieve this condition by the following rule:

$\forall x Q \rightsquigarrow \forall y Q[y/x]$
Here, $Q[y/x]$ is Q with all free occurrences of x replaced by y.

Prenex Form: Rules for negations, implications, equivalences

$$\neg\forall x P \rightsquigarrow \exists x \neg P \qquad \neg\exists x P \rightsquigarrow \forall x \neg P$$

$$\forall x Q \to P \rightsquigarrow \exists x (Q \to P) \qquad \exists x Q \to P \rightsquigarrow \forall x (Q \to P)$$

$$P \to \forall x Q \rightsquigarrow \forall x (P \to Q) \qquad P \to \exists x Q \rightsquigarrow \exists x (P \to Q)$$

$$P \leftrightarrow Q \rightsquigarrow (P \to Q) \wedge (Q \to P)$$

Note that for the second and third line, x must not be a free variable in P.

Prenex Form: example

What is the prenex normal form of

$$\exists x Cube(x) \to \forall y Small(y)$$

$$\forall x[(Cube(x) \wedge \exists y(Tet(y) \wedge LeftOf(x, y))) \to \exists y(Dodec(y) \wedge BackOf(x, y))]$$

Proof methods for quantifiers

Proof methods for quantifiers

Universal elimination

Universal statments can be instantiated to any object.

From $\forall x S(x)$, we may infer $S(c)$.

Existential introduction

If we have established a statement for an instance, we can also establish the corresponding existential statement.

From $S(c)$, we may infer $\exists x S(x)$.

Example

$$\forall x[\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)]$$
$$\forall x[\mathrm{Large}(x) \rightarrow \mathrm{LeftOf}(x, b)]$$
$$\mathrm{Cube}(d)$$

$$\exists x[\mathrm{Large}(x) \wedge \mathrm{LeftOf}(x, b)]$$

Existential elimination

From $\exists x S(x)$, we can infer things by assuming $S(c)$ in a subproof, if c is a new name not used otherwise.
Example: Scotland Yard searched a serial killer. The did not know who he was, but for their reasoning, they called him "Jack the ripper".
This would have been an unfair procedure if there had been a real person named Jack the ripper.

Example

$$\forall x[\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)]$$
$$\forall x[\mathrm{Large}(x) \rightarrow \mathrm{LeftOf}(x, b)]$$
$$\exists x \mathrm{Cube}(x)$$

$$\exists x[\mathrm{Large}(x) \wedge \mathrm{LeftOf}(x, b)]$$