Logik für Informatiker
Logic for computer scientists

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Truth-functional form

- Replace all top-level quantified sub-formulas (i.e. those not occurring below another quantifier) by propositional letters.
- Replace multiple occurrences of the same sub-formula by the same propositional letter.
- A quantified sentence of FOL is said to be a tautology iff its truth-functional form is a tautology.

$$\forall x\ Cube(x) \lor \neg\forall x\ Cube(x)$$

becomes

$$A \lor \neg A$$

Truth functional form — examples

FO sentence	t.f. form
$\forall x\, Cube(x) \lor \neg \forall x\, Cube(x)$	$A \lor \neg A$
$(\exists y\, Tet(y) \land \forall z\, Small(z)) \to \forall z\, Small(z)$	$(A \land B) \to B$
$\forall x\, Cube(x) \lor \exists y\, Tet(y)$	$A \lor B$
$\forall x\, Cube(x) \to Cube(a)$	$A \to B$
$\forall x\, (Cube(x) \lor \neg Cube(x))$	A
$\forall x\, (Cube(x) \to Small(x)) \lor \exists x\, Dodec(x)$	$A \lor B$

Examples of $\rightarrow$-Elim

$\exists x(\mathrm{Cube}(x) \rightarrow \mathrm{Small}(x))$
$\exists x\ \mathrm{Cube}(x)$

$\exists x\ \mathrm{Small}(x)$

No!

A
B

C

$\exists x \mathrm{Cube}(x) \rightarrow \exists x\ \mathrm{Small}(x)$
$\exists x\ \mathrm{Cube}(x)$

$\exists x\ \mathrm{Small}(x)$

Yes!

$A \rightarrow B$
A

B

Tautologies and logical truths

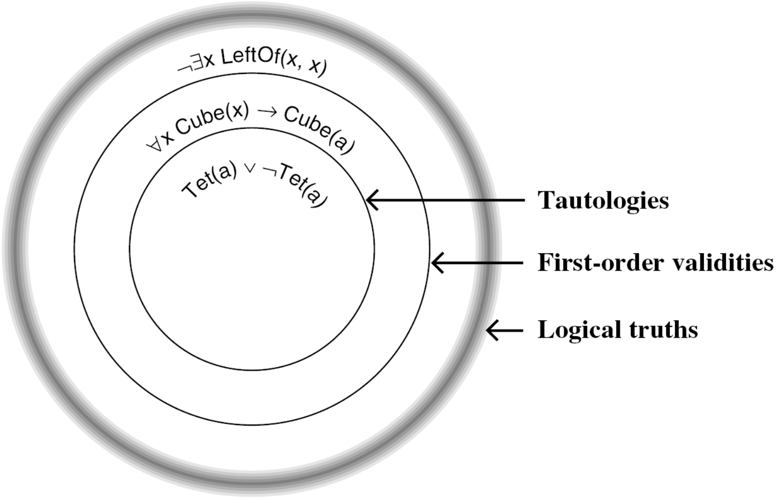
Every tautology is a logical truth, but not vice versa.
Example: $\exists x\ Cube(x) \lor \exists x\ \neg Cube(x)$
is a logical truth, but not a tautology.
Similarly, every tautologically valid argument is a logically valid argument, but not vice versa.

$\forall$x Cube(x)

$\exists$x Cube(x)

is a logically valid argument, but not tautologically valid.

Different notions of validity



Tautologies and logical truths, cont'd

Propositional logic	First-order logic	Tarski' World	General notion
Tautology	*FO validity*	*TW validity*	*Logical Truth*
Tautological consequence	*FO consequence*	*TW consequence*	*Logical consequence*
Tautological equivalence	*FO equivalence*	*TW equivalence*	*Logical equivalence*

Which ones are FO validities?

$$\forall x\ SameSize(x,x)$$
$$\forall x\ Cube(x) \to Cube(b)$$
$$(Cube(b) \land b = c) \to Cube(c)$$
$$(Small(b) \land SameSize(b,c)) \to Small(c)$$

Replacement method: Replace predicates by meaningless ones

$$\forall x \; Outgrabe(x, x)$$
$$\forall x \; Tove(x) \rightarrow Tove(b)$$
$$(Tove(b) \wedge b = c) \rightarrow Tove(c)$$
$$(Slithy(b) \wedge Outgrabe(b, c)) \rightarrow Slithy(c)$$

Is this a valid FO argument?

$$\begin{array}{|l} \forall x(\mathrm{Tet}(x) \rightarrow \mathrm{Large}(x)) \\ \neg \mathrm{Large}(b) \\ \hline \neg \mathrm{Tet}(b) \end{array}$$

Replacement with nonsense predicates:

$$\begin{array}{|l} \forall x(\mathrm{Borogove}(x) \rightarrow \mathrm{Mimsy}(x)) \\ \neg \mathrm{Mimsy}(b) \\ \hline \neg \mathrm{Borogove}(b) \end{array}$$

Is this a valid FO argument?

Replacement with a meaningless predicate:

$\neg\exists x\ \mathsf{Larger}(x, a)$	$\neg\exists x\ R(x, a)$
$\neg\exists x\ \mathsf{Larger}(b, x)$	$\neg\exists x\ R(b, x)$
$\mathsf{Larger}(c, d)$	$R(c, d)$
$\mathsf{Larger}(a, b)$	$R(a, b)$

A counterexample

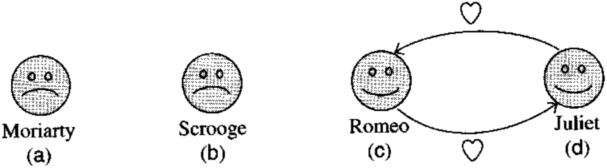


Figure 10.1: A first-order counterexample.

First-order equivalence

Two well-formed formulas P and Q (possibly containing free variables) are logically equivalent, if in all circumstances, they are satisfied by the same objects. This is written as

$$P \Leftrightarrow Q$$

Substitution principle

If $P \Leftrightarrow Q$, then $S(P) \Leftrightarrow S(Q)$.

Here, $S(\_)$ is a sentence with a "hole".

DeMorgan laws for quantifiers

$$\begin{array}{ccc} \neg\forall x P(x) & \Leftrightarrow & \exists x \neg P(x) \\ \forall x P(x) & \Leftrightarrow & \neg\exists x \neg P(x) \end{array}$$

$$\begin{array}{ccc} \neg\exists x P(x) & \Leftrightarrow & \forall x \neg P(x) \\ \exists x P(x) & \Leftrightarrow & \neg\forall x \neg P(x) \end{array}$$

More quantifier equivalences

$$\forall x(P(x) \wedge Q(x)) \quad \Leftrightarrow \quad \forall x P(x) \wedge \forall x Q(x)$$
$$\forall x(P(x) \vee Q(x)) \quad \not\Leftrightarrow \quad \forall x P(x) \vee \forall x Q(x)$$

$$\exists x(P(x) \vee Q(x)) \quad \Leftrightarrow \quad \exists x P(x) \vee \exists x Q(x)$$
$$\exists x(P(x) \wedge Q(x)) \quad \not\Leftrightarrow \quad \exists x P(x) \wedge \exists x Q(x)$$

$$\left.\begin{array}{ccc} \forall x P & \Leftrightarrow & P \\ \exists x P & \Leftrightarrow & P \\ \forall x(P \vee Q(x)) & \Leftrightarrow & P \vee \forall x Q(x) \\ \exists x(P \wedge Q(x)) & \Leftrightarrow & P \wedge \exists x Q(x) \end{array}\right\} \text{ if } x \text{ is not a free variable in } P$$

TW consequence $\neq$ FO consequence

We have encountered arguments that are valid in Tarski's World but not FO valid.

$\forall x(\text{Cube}(x) \leftrightarrow \text{SameShape}(x, c))$

$\text{Cube}(c)$

The replacement method yields an invalid argument:

$\forall x(P(x) \leftrightarrow Q(x, c))$

$P(c)$

The axiomatic method

Axiomatic method: bridge the gap between Tarski's World validity and FO validity by systematically expressing facts about the meanings of the predicates, and introduce them as *axioms*. Axioms restrict the possible interpretation of predicates.
Axioms may be used as premises within arguments/proofs.

The argument revisited

$$\begin{array}{|l} \forall x(\mathsf{Cube}(x) \leftrightarrow \mathsf{SameShape}(x,c)) \\ \forall x \mathsf{SameShape}(x,x) \\ \hline \mathsf{Cube}(c) \end{array}$$

The replacement method yields a valid argument:

$$\begin{array}{|l} \forall x(P(x) \leftrightarrow Q(x,c)) \\ \forall x Q(x,x) \\ \hline P(c) \end{array}$$

The basic shape axioms

1. $\neg\exists x(Cube(x) \wedge Tet(x))$
2. $\neg\exists x(Tet(x) \wedge Dodec(x))$
3. $\neg\exists x(Dodec(x) \wedge Cube(x))$
4. $\forall x(Tet(x) \vee Dodec(x) \vee Cube(x))$

An argument using the shape axioms

$$\begin{array}{|l} \neg\exists x(\mathrm{Dodec}(x) \land \mathrm{Cube}(x)) \\ \forall x(\mathrm{Tet}(x) \lor \mathrm{Dodec}(x) \lor \mathrm{Cube}(x)) \\ \neg\exists x\ \mathrm{Tet}(x) \\ \hline \forall x(\mathrm{Cube}(x) \leftrightarrow \neg\mathrm{Dodec}(x)) \end{array}$$

$$\begin{array}{|l} \neg\exists x(P(x) \land Q(x)) \\ \forall x(R(x) \lor P(x) \lor Q(x)) \\ \neg\exists x\ R(x) \\ \hline \forall x(Q(x) \leftrightarrow \neg P(x)) \end{array}$$

SameShape introduction and elimination axioms

1. $\forall x \forall y((Cube(x) \land Cube(y)) \rightarrow SameShape(x, y))$
2. $\forall x \forall y((Dodec(x) \land Dodec(y)) \rightarrow SameShape(x, y))$
3. $\forall x \forall y((Tet(x) \land Tet(y)) \rightarrow SameShape(x, y))$

4. $\forall x \forall y((SameShape(x, y) \land Cube(x)) \rightarrow Cube(y))$
5. $\forall x \forall y((SameShape(x, y) \land Dodec(x)) \rightarrow Dodec(y))$
6. $\forall x \forall y((SameShape(x, y) \land Tet(x)) \rightarrow Tet(y))$

Euclid's axiomatization of geometry

1. Any two points can be joined by a straight line.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. Parallel postulate. If two lines intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

Peano's axiomatization of the naturals

1. 0 is a natural number.
2. For every natural number, its successcor is a natural number.
3. There is no natural number whose successor is 0.
4. Two different natural numbers have different successors.
5. If K is a set such that:
    - 0 is in K, and
    - for every natural number in K, its successor also is in K,

then K contains every natural number.

Formalization of Peano's axioms

1. a constant 0
2. a unary function symbol *suc*
3. $\forall n\ \neg suc(n) = 0$
4. $\forall m \forall n\ suc(m) = suc(n) \rightarrow m = n$
5. $(\Phi(x/0) \wedge \forall n(\Phi(x/n) \rightarrow \Phi(x/suc(n)))) \rightarrow \forall n\ \Phi(x/n)$
   if Φ is a formula with a free variable x, and
   $\Phi(x/t)$ denotes the replacement of x with t within Φ

Other famous axiom systems

- Zermelo-Fraenkel axiomatization of set theory
- axiomatizations in algebra: monoids, groups, rings, fields, vector spaces . . .
- Hoare's axiomatization of imperative programming with while-loops, if-then-else and assignment