Logik für Informatiker
Logic for computer scientists

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Soundness and Completeness

Tautological consequence

A sentence S is a tautological consequence of a set of sentences $\mathcal{T}$, written

$$\mathcal{T} \models_T S,$$

if all valuations of atomic formulas with truth values that make all sentences in $\mathcal{T}$ true also make S true.

Propositional proofs

S is $\mathcal{F}_T$-provable from $\mathcal{T}$, written

$$\mathcal{T} \vdash_T S,$$

if there is a formal proof of S with premises drawn from $\mathcal{T}$ using the elimination and introduction rules for $\vee, \wedge, \neg, \rightarrow, \leftrightarrow$ and $\bot$.

Again note: $\mathcal{T}$ may be infinite.

Soundness

Theorem 1. The proof calculus $\mathcal{F}_T$ is sound, i.e. if

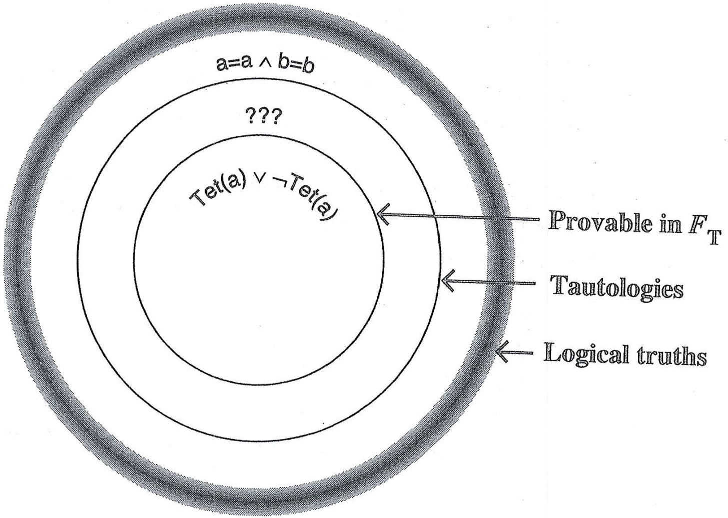
$$\mathcal{T} \vdash_T S,$$

then

$$\mathcal{T} \models_T S.$$

Proof: Book: by contradiction, using the first invalid step.
Here: by induction on the length of the proof.

Soundness



Completeness

Theorem 2 (Bernays, Post). The proof calculus $\mathcal{F}_T$ is complete, i.e. if

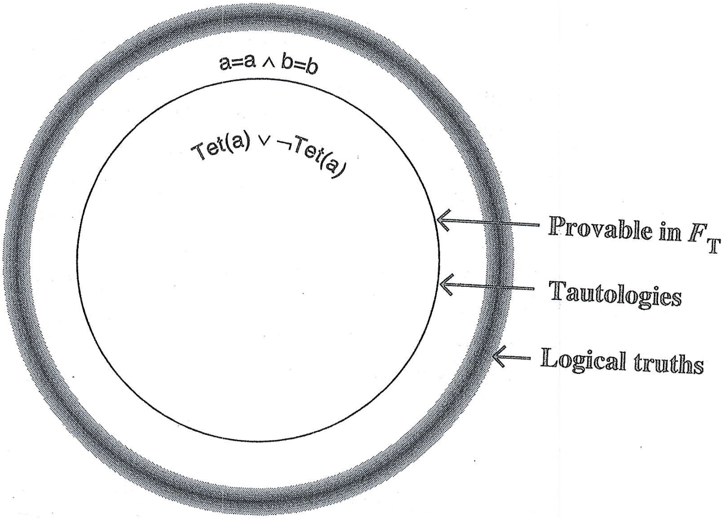
$$\mathcal{T} \models_T S,$$

then

$$\mathcal{T} \vdash_T S.$$

This theorem will be proved later in the lecture.

Completeness



Quantifiers

Quantifiers: Motivating examples

$$\frac{\begin{array}{l}\forall x\ Cube(x)\ \text{(“All objects are cubes.”)}\\ \forall x\ (Cube(x) \rightarrow Large(x))\ \text{(“All cubes are large.”)}\end{array}}{\forall x\ Large(x)\ \text{(“All objects are large.”)}}$$

$$\exists x\ Cube(x)$$

“There exists a cube.”

$$\exists x\ (Cube(x) \wedge Large(x))$$

“There exists a large cube.”

The four Aristotelian forms

All P's are Q's.	$\forall x(P(x) \rightarrow Q(x))$
Some P's are Q's.	$\exists x(P(x) \wedge Q(x))$
No P's are Q's.	$\forall x(P(x) \rightarrow \neg Q(x))$
Some P's are not Q's.	$\exists x(P(x) \wedge \neg Q(x))$

Note:
$\forall x(P(x) \rightarrow Q(x))$ does not imply that there are some $P's$.
$\exists x(P(x) \wedge Q(x))$ does not imply that not all $P's$ are $Q's$.

First-order language

A first-order language consists of

- a set of predicate symbols with arities, like $\mathit{Smaller}^{(2)}$, $\mathit{Dodec}^{(1)}$, $\mathit{Between}^{(3)}$, $\leq^{(2)}$, including propositional symbols (nullary predicate symbols), like $A^{(0)}$, $B^{(0)}$, $C^{(0)}$, (written uppercase)
- its names or constants for individuals, like a, b, c, (written lowercase)
- its function symbols with arities, like $f^{(1)}$, $+^{(2)}$, $\times^{(2)}$.

Usually, arities are omitted.

Well-formed terms

$t ::= a$	individual constant
$\mid x$	variable
$\mid f^{(n)}(t_1, \ldots, t_n)$	application of function symbols to terms $t_1, \ldots, t_n$ (recursive definition)

Usually, arities are omitted.
Variables are: t, u, v, w, x, y, z, possibly with subscripts.
Individual constants are: a, b, c, d, e, f, n, and others.

Well-formed formulas

$F ::= P^{(n)}(t_1, \ldots, t_n)$	application of predicate symbols to terms $t_1, \ldots, t_n$
$\mid t_1 = t_2$	equality
$\mid \bot$	contradiction
$\mid \neg F$	negation
$\mid (F_1 \wedge \ldots \wedge F_n)$	conjunction
$\mid (F_1 \vee \ldots \vee F_n)$	disjunction
$\mid (F_1 \rightarrow F_2)$	implication
$\mid (F_1 \leftrightarrow F_2)$	equivalence
$\mid \forall \nu F$	universal quantification
$\mid \exists \nu F$	existential quantification

The variable ν is said to be bound in $\forall \nu F$ and $\exists \nu F$.

Here, F, and $F_1 \ldots F_n$ occurring on the right-hand side are formulas again (recursive definition).

Parentheses

The outermost parentheses of a well-formed formula can be omitted:

$$Cube(x) \land Small(x)$$

In general, parentheses are important to determine the scope of a quantifier (see next slide).

Free and bound variables

An occurrence of a variable in a formula that is not bound (by a quantifier) is said to be free.

$\exists y\ LeftOf(x, y)$	x is free, y is bound
$(Cube(x) \land Small(x)) \rightarrow \exists y\ LeftOf(x, y)$	x is free, y is bound
$\exists x\ (Cube(x) \land Small(x))$	Both occurrences of x are bound
$\exists x\ Cube(x) \land Small(x)$	The first occurrence of x is bound, the second one is free

Sentences

A sentence is a well-formed formula without free variables.

$$\bot \qquad\qquad A \wedge B$$

$$Cube(a) \vee Tet(b)$$

$$\forall x\ (Cube(x) \rightarrow Large(x))$$

$$\forall x\ ((Cube(x) \wedge Small(x)) \rightarrow \exists y\ LeftOf(x, y))$$

Semantics of quantification

- We need to fix some domain of discourse.
- $\forall x\ S(x)$ is true iff for every object in the domain of discourse with name n, $S(n)$ is true.
- $\exists x\ S(x)$ is true iff for some object in the domain of discourse with name n, $S(n)$ is true.
- Not all objects need to have names — hence we assume that for objects, names $n_1, n_2, \ldots$ can be invented "on the fly".

N.B. "iff" means "if and only if" and expresses (meta) equivalence

The game rules

Form	Your commitment	Player to move	Goal
$P \vee Q$	TRUE FALSE	you Tarski's World	Choose one of P, Q that is true.
$P \wedge Q$	TRUE FALSE	Tarski's World you	Choose one of P, Q that is false.
$\exists x\, P(x)$	TRUE FALSE	you Tarski's World	Choose some ***b*** that satisfies the wff P(x).
$\forall x\, P(x)$	TRUE FALSE	Tarski's World you	Choose some ***b*** that does not satisfy P(x).

Logical consequence for quantifiers

$$\begin{array}{|l} \forall x(\mathrm{Cube}(x) \rightarrow \mathrm{Small}(x)) \\ \forall x\ \mathrm{Cube}(x) \\ \hline \forall x\ \mathrm{Small}(x) \end{array}$$

$$\begin{array}{|l} \forall x\ \mathrm{Cube}(x) \\ \forall x\ \mathrm{Small}(x) \\ \hline \forall x(\mathrm{Cube}(x) \wedge \mathrm{Small}(x)) \end{array}$$

However: ignoring quantifiers does not work!

$$\begin{array}{|l} \exists x(\mathrm{Cube}(x) \rightarrow \mathrm{Small}(x)) \\ \exists x\ \mathrm{Cube}(x) \\ \hline \exists x\ \mathrm{Small}(x) \end{array}$$

$$\begin{array}{|l} \exists x\ \mathrm{Cube}(x) \\ \exists x\ \mathrm{Small}(x) \\ \hline \exists x(\mathrm{Cube}(x) \land \mathrm{Small}(x)) \end{array}$$

Tautologies do not distribute over quantifiers

$$\exists x\ Cube(x) \lor \exists x\ \neg Cube(x)$$

is a logical truth, but

$$\forall x\ Cube(x) \lor \forall x\ \neg Cube(x)$$

is not. By contrast,

$$\forall x\ Cube(x) \lor \neg\forall x\ Cube(x)$$

is a tautology.

Truth-functional form

Replace all top-level quantified sub-formulas (i.e. those not ocurring below another quantifier) by propositional letters.
Replace multiple occurrences of the same sub-formula by the same propositional letter.
A quantified sentence of FOL is said to be a tautology iff its truth-functional form is a tautology.

$$\forall x\ Cube(x) \lor \neg\forall x\ Cube(x)$$

becomes

$$A \lor \neg A$$

Truth functional form — examples

FO sentence	t.f. form
$\forall x\, Cube(x) \lor \neg \forall x\, Cube(x)$	$A \lor \neg A$
$(\exists y\, Tet(y) \land \forall z\, Small(z)) \rightarrow \forall z\, Small(z)$	$(A \land B) \rightarrow B$
$\forall x\, Cube(x) \lor \exists y\, Tet(y)$	$A \lor B$
$\forall x\, Cube(x) \rightarrow Cube(a)$	$A \rightarrow B$
$\forall x\, (Cube(x) \lor \neg Cube(x))$	A
$\forall x\, (Cube(x) \rightarrow Small(x)) \lor \exists x\, Dodec(x)$	$A \lor B$