Logik für Informatiker
Logic for computer scientists

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The language of PL1

The language of PL1: individual constants

- Individual constants are symbols that denote a person, thing, object
- Examples:
  - Numbers: 0, 1, 2, 3, . . .
  - Names: Max, Claire
  - Formal constants: a, b, c, d, e, f, n1, n2
- Each individual constant must denote an existing object
- No individual constant can denote more than one object
- An object can have 0, 1, 2, 3 . . . names

The language of PL1: predicate symbols

- Predicate symbols denote a property of objects, or a relation between objects
- Each predicate symbol has an arity that tell us how many objects are related
- Examples:
  - Arity 0: Gate0_is_low, A, B, ...
  - Arity 1: Cube, Tet, Dodec, Small, Medium, Large
  - Arity 2: Smaller, Larger, LeftOf, BackOf, SameSize, Adjoins ...
  - Arity 3: Between

The interpretation of predicate symbols

- In Tarski's world, predicate symbols have a fixed interpretation, that not always completely coindices with the natural language interpretation
- In other PL1 languages, the interpretation of predicate symbols may vary. For example, $\leq$ may be an ordering of numbers, strings, trees etc.
- Usually, the binary symbol $=$ has a fixed interpretation: equality

Atomic sentences

- in propositional logic (Boole:)
  - propositional symbols: Gate0_is_low, A, B, C, ...
- in PL1 (Tarski's world):
  - application of predicate symbols to constants: Larger(a,b)
  - the order of arguments matters: Larger(a,b) vs. Larger(b,a)
  - Atomic sentences denote truth values (true, false)

Logical arguments

A (logical) argument states that a sentence, the conclusion, follows from other sentences, the premises.

Examples:

All men are mortal. Socrates is a man. So, Socrates is mortal.

Lucretius is a man. After all, Lucretius is mortal and all men are mortal.

An argument is valid (or a logical consequence), if truth is preserved, that is, all circumstances that make the premises true, also make the conclusion true.

Logical consequence

A sentence B is a logical consequence of A_1, ..., A_n, if all circumstances that make A_1, ..., A_n true also make B true.
In symbols: $A_1, \ldots, A_n \models B$.

In this case, it is a valid argument to infer B from $A_1, \ldots A_n$. If also $A_1, \ldots A_n$ are true, then the valid argument is sound.

$A_1, \ldots, A_n$ are called premises, B is called conclusion.

Logical consequence — examples

- All men are mortal. Socrates is a man. So, Socrates is mortal. (valid, sound)
- All rich actors are good actors. Brad Pitt is a rich actor. So he must be a good actor. (valid, but not sound)
- All rich actors are good actors. Brad Pitt is a good actor. So he must be a rich actor. (not valid)

Fitch notation for logical consequence

All men are mortal
Socrates is a man

So, Socrates is mortal

A_1
. . .
A_n

B

Premise_1
. . .
Premise_n

Conclusion

Methods for showing (in)validity of arguments

Methods for showing (in)validity of arguments

Validity To show that an argument is valid, we must provide a proof. A proof consists of a sequence of proof steps, each of which must be valid.

- In propositional logic, we also can use truth tables to show validity. This is not possible in first-order logic.

Invalidity An argument can shown to be invalid by finding a counterexample (model), i.e. a circumstance where the premises are true, but the conclusion is false.

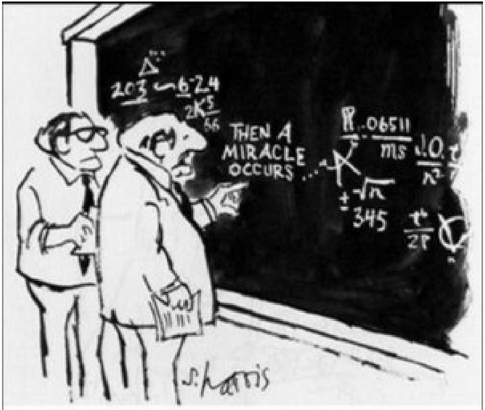
Informal and formal proofs

- informal reasoning is used in everyday life
- semi-formal reasoning is used in mathematics and theoretical computer science
  - balance between readability and precision
- formal proofs:
  - follow some specific rule system,
  - and are entirely rigorous
  - and can be checked by a computer

An informal proof

- Since Socrates is a man and all men are mortal, it follows that Socrates is mortal.
- But all mortals will eventually die, since that is what it means to be mortal.
- So Socrates will eventually die.
- But we are given that everyone who will eventually die sometimes worries about it.
- Hence Socrates sometimes worries about dying.

The need for formal proofs



"I think you should be more explicit here in step two."

A formal proof

1. Cube(c)
2. c = b

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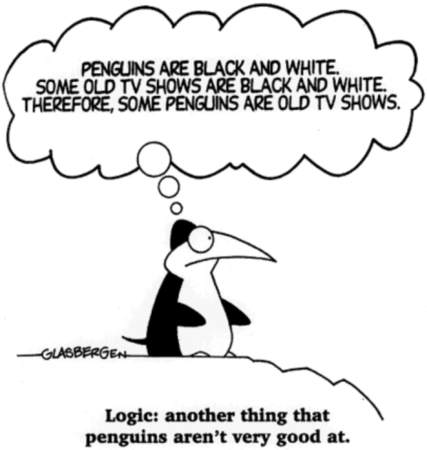
3. Cube(b) **=Elim:** 1,2

Four principles for the identity relation

1. $=$**Elim**: If $b = c$, then whatever holds of b holds of c (indiscernibility of identicals).
2. $=$**Intro**: $b = b$ is always true in FOL (reflexivity of identity).
3. Symmetry of Identity: If $b = c$, then $c = b$.
4. Transitivity of Identity: If $a = b$ and $b = c$, then $a = c$.

The latter two principles follow from the first two.

Transitivity . . .



Informal proof of symmetry of identity

- Suppose that $a = b$.
- We know that $a = a$, by the reflexivity of identity.
- Now substitute the name b for the first use of the name a in $a = a$, using the indiscernibility of identicals.
- We come up with $b = a$, as desired.

Formal Proofs in Fitch

Formal proofs

P	
Q	
R	
S_1	Justification 1
...	
...	
S_n	Justification n
S	Justification n+1

Formal proof of symmetry of identity

1. a = b	
2. a = a	= **Intro:**
3. b = a	= **Elim:** 2,1

Fitch rule: Identity introduction

Identity Introduction (= Intro):

$$\triangleright \mid \; \mathsf{n} = \mathsf{n}$$

Fitch rule: Identity elimination

Identity Elimination (= Elim):

$$\begin{array}{l|l} & P(n) \\ & \vdots \\ & n = m \\ & \vdots \\ \triangleright & P(m) \end{array}$$

Fitch rule: Reiteration

Reiteration (Reit):

$$
\begin{array}{l|l}
 & P \\
 & \vdots \\
\triangleright & P
\end{array}
$$

Example proof in fitch

$$\begin{array}{|l} \text{SameRow}(a, b) \\ b = a \\ \hline \text{SameRow}(b, a) \end{array}$$

Properties of predicates in Tarski's world

$$\begin{array}{|l} \text{Larger}(a, b) \\ \text{Larger}(b, c) \\ \hline \text{Larger}(a, c) \end{array}$$

$$\begin{array}{|l} \text{RightOf}(b, c) \\ \hline \text{LeftOf}(c, b) \end{array}$$

Such arguments can be proved in Fitch using the special rule **Ana Con**.

This rule is only valid for reasoning about Tarski's world!

Showing invalidity using counterexamples

Al Gore is a politician
Hardly any politicians are honest

Al Gore is dishonest

Imagine a situation where there are 10,000 politicians, and that Al Gore is the only honest one of the lot. In such circumstances both premises would be true but the conclusion would be false.
This demonstrates that the argument is invalid.

Are the following arguments valid?

| Small(a)
| Larger(b, a)
|—
| Large(b)

| Small(a)
| Larger(a, b)
|—
| Large(b)