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Alphabets, Words, Languages I

alphabet — non-empty finite set
letter — element of an alphabet

word (over V) — sequence of letters (of V)
λ — empty word

V^* (V^+, resp.) — set of all (non-empty) words over V

product (concatenation) of words — juxtaposition of words

v subword of w — $w = x_1 v x_2$ for some $x_1, x_2 \in V^*$
v prefix of w — $w = vx$ for some $x \in V^*$
v suffix of w — $w = xv$ for some $x \in V^*$

Alphabets, Words, Languages II

$\#_a(w)$ — number of occurrences of the letter a in the word w
$|w| = \sum_{a\in V} \#_a(w)$ — length of the word $w \in V^*$
$\Psi(w) = (\#_{a_1}(w), \#_{a_2}(w), \ldots, \#_{a_n}(w))$ — Parikh vector of the word $w \in V^*$, $V = \{a_1, a_2, \ldots, a_n\}$

language (over V) — subset (of V^*)

Convention: languages L_1 and L_2 are called equal (written as $L_1 = L_2$) if and only if L_1 and L_2 differ at most in the empty word
i.e., $L_1 \setminus \{\lambda\} = L_2 \setminus \{\lambda\}$

Grammars I

phrase structure grammar — $G = (N, T, P, S)$,
N – set of nonterminals,
T – set of terminals
$V_G = N \cup T$, $N \cap T = \emptyset$,
P – set of productions/rules,
finite subset of $(V^* \setminus T^*) \times V^*)$,
rules written as $\alpha \to \beta$,
$S \in N$ – axiom

direct derivation $x \Longrightarrow_G y$ — $x = x_1\alpha x_2$, $y = x_1\beta x_2$, $\alpha \to \beta \in P$,

$\Longrightarrow_G^*$ reflexive and transitive closure of $\Longrightarrow_G$

$L(G) = \{z \mid z \in T^*$ and $S \Longrightarrow_G^* z\}$

Grammars II

G is called monotone if and only if every rule of P has the form $\alpha \rightarrow \beta$ with $|\alpha| \leq |\beta|$,

G is called context-sensitive if and only if every rule of P has the form $uAv \rightarrow uwv$ with $A \in N$, $w \in V^+$, $u, v \in V^*$

G is called context-free if and only if every rule of P has the form $A \rightarrow w$ with $A \in N$ and $w \in V^*$,

G is called regular if and only if every rule of P has the form $A \rightarrow wB$ or $A \rightarrow w$ with $A, B \in N$ and $w \in T^*$,

Grammars III

REG, CF, CS, MON and RE — families of regular, context-free, context-sensitive, monotone and arbitrary (phrase structure) grammars,

language L is called X if and only if $L = L(G)$ for some X grammar G

for a family X of grammar,
$\mathcal{L}(X)$ — family of languages generated by grammars of X,
$\mathcal{L}(FIN)$ — family of finite languages

$\mathcal{L}(FIN) \subset \mathcal{L}(REG) \subset \mathcal{L}(CF) \subset \mathcal{L}(CS) = \mathcal{L}(MON) \subset \mathcal{L}(RE)$
$\mathcal{L}(RE)$ is the family of all recursively enumerable languages

Workspace of Grammars

$G = (N, T, P, S)$ — phrase structure grammar

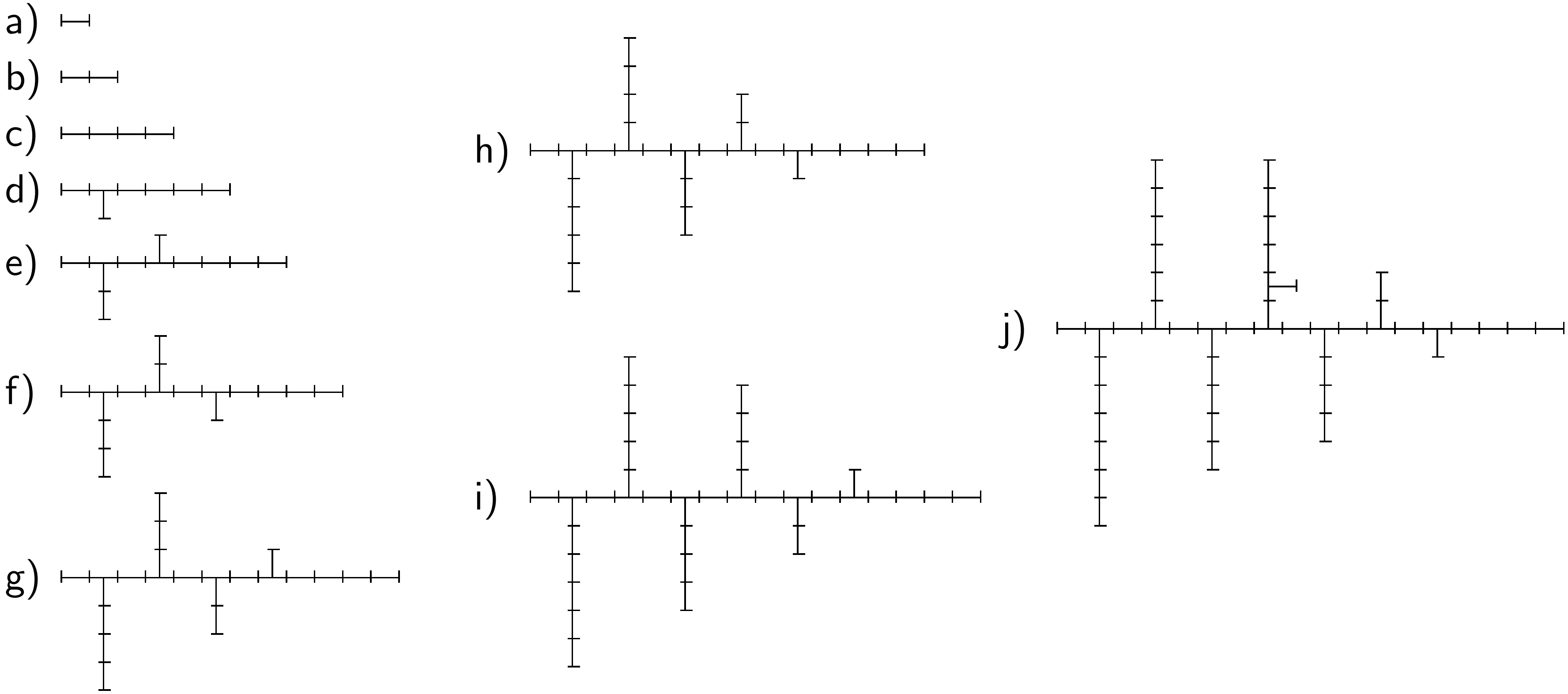
$D : S \Longrightarrow w_1 \Longrightarrow w_2 \Longrightarrow \ldots \Longrightarrow w_r = w$ — derivation of $w \in T^*$ in G

workspace of w by D — $Ws_G(w, d) = max\{|w_i| \mid 1 \leq i \leq r\}$

workspace of w — $Ws_G(w) = \min\{Ws_G(w, D) \mid D$ is a derivation of w in $G\}$,

Workspace Theorem:
If $G = (N, T, P, S)$ is a phrase structure grammar and k is a positive integer such that $Ws_G(w) \leq k|w|$ holds for any $w \in L(G)$, then $L(G)$ is a context-sensitive language.

Development of some Alga I



Development of some Alga II

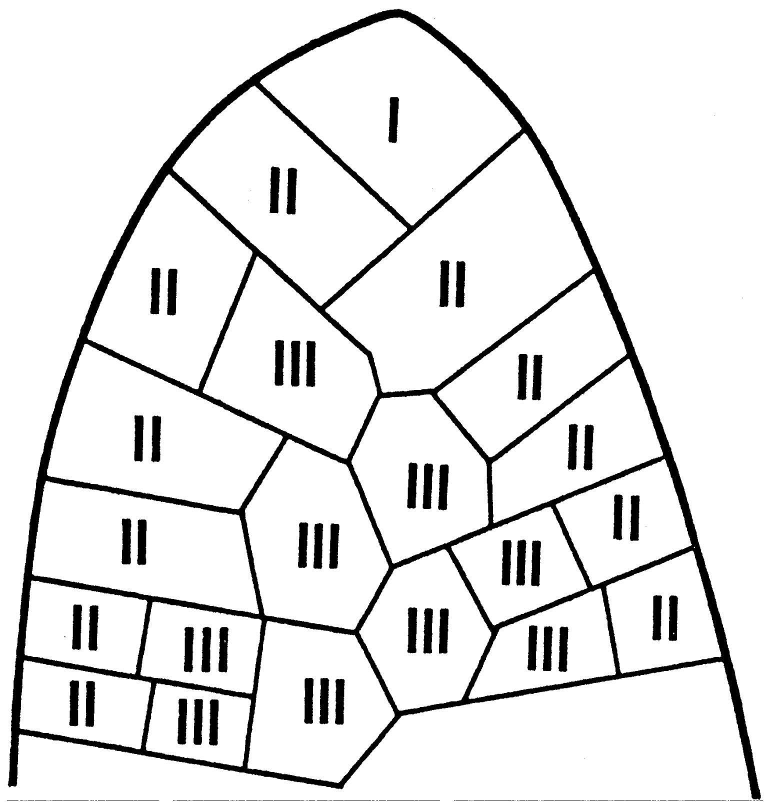
a) c
b) cc
c) $cccc$
d) $cc(c)cccc$
e) $cc(cc)cc(c)cccc$
f) $cc(ccc)cc(cc)cc(c)cccc$
g) $cc(cccc)cc(ccc)cc(cc)cc(c)cccc$
h) $cc(ccccc)cc(cccc)cc(ccc)cc(cc)cc(c)cccc$
i) $cc(cccccc)cc(ccccc)cc(cccc)cc(cccc)cc(cc)cc(c)cccc$
j) $cc(ccccccc)cc(cccccc)cc(ccccc)cc(cc(c)cccc)cc(cccc)cc(cc)cc(c)cccc$

Development of some Alga III

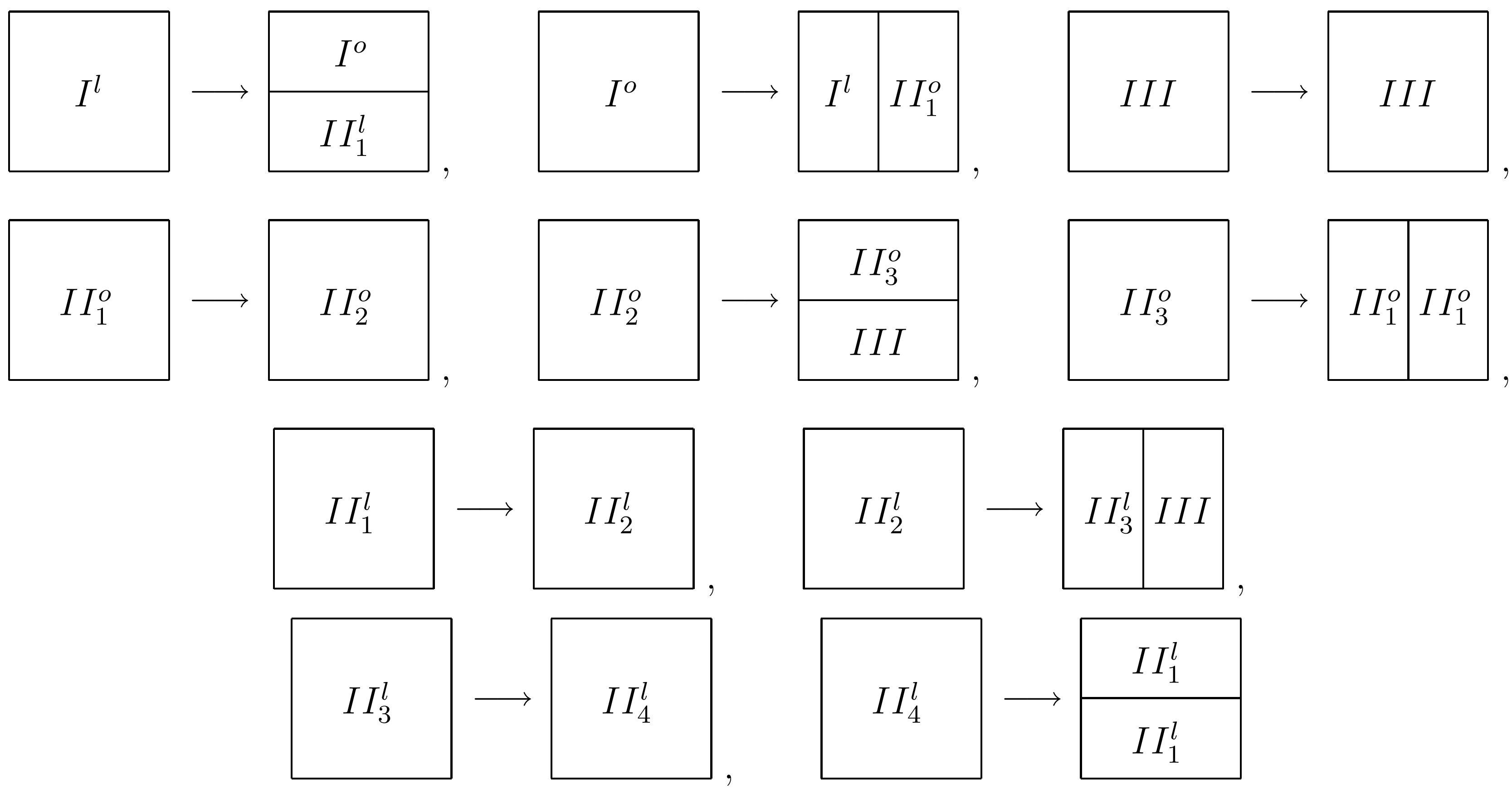
$$0 \to 10 \quad 1 \to 32 \quad 2 \to 3(4) \quad 3 \to 3 \quad 4 \to 56$$
$$5 \to 37 \quad 6 \to 58 \quad 7 \to 3(9) \quad 8 \to 50 \quad 9 \to 39$$

a) 4
b) 56
c) 3758
d) $33(9)3750$
e) $33(39)33(9)3710$
f) $33(339)33(39)33(9)3210$
g) $33(3339)33(339)33(39)33(4)3210$
h) $33(33339)33(3339)33(339)33(56)33(4)3210$
i) $33(333339)33(33339)33(3339)33(3758)33(56)33(4)3210$
j) $33(3333339)33(333339)33(33339)33(33(9)3750)33(3758)33(56)33(4)3210$

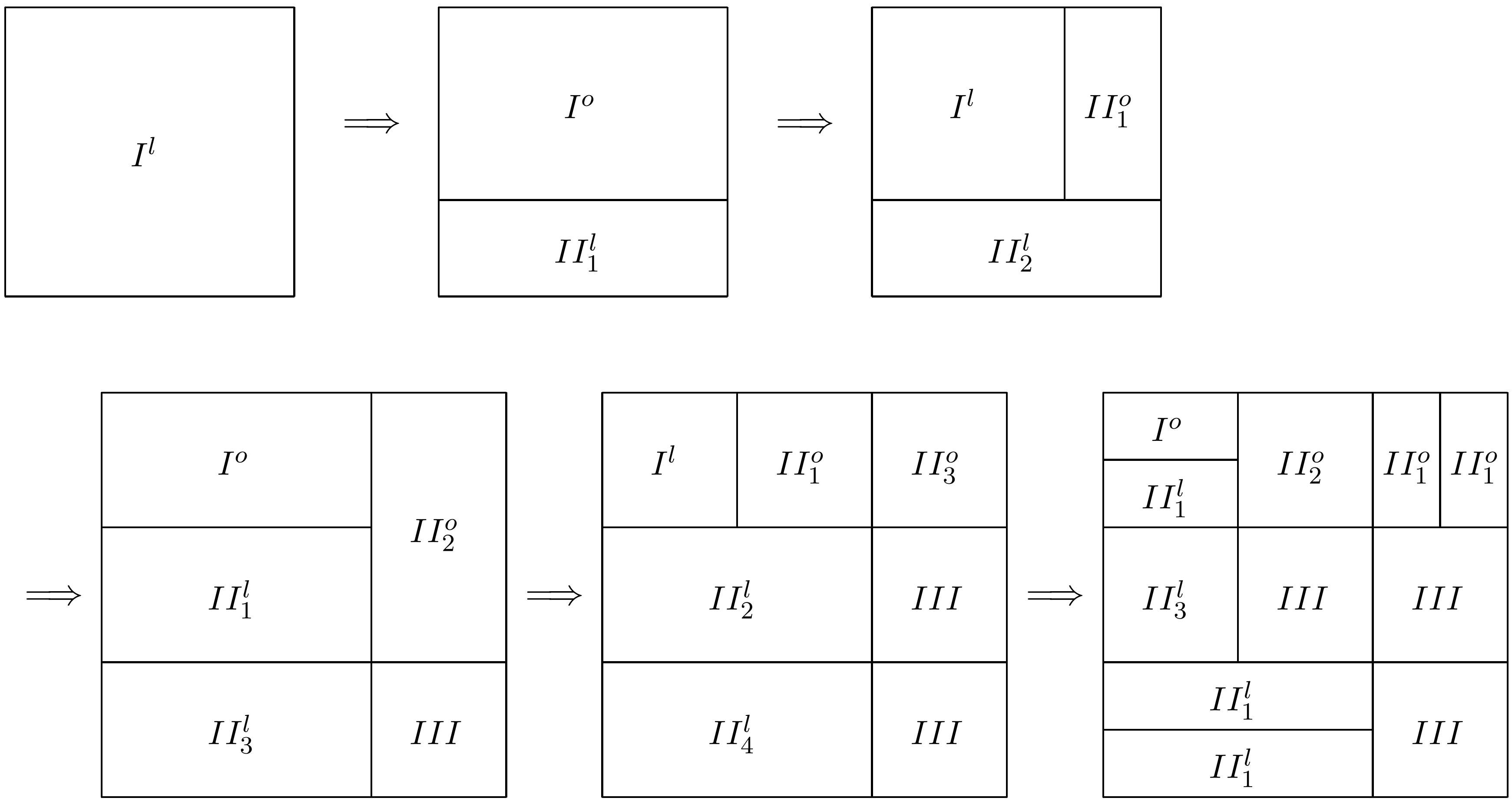
Phascum Cuspidatum I



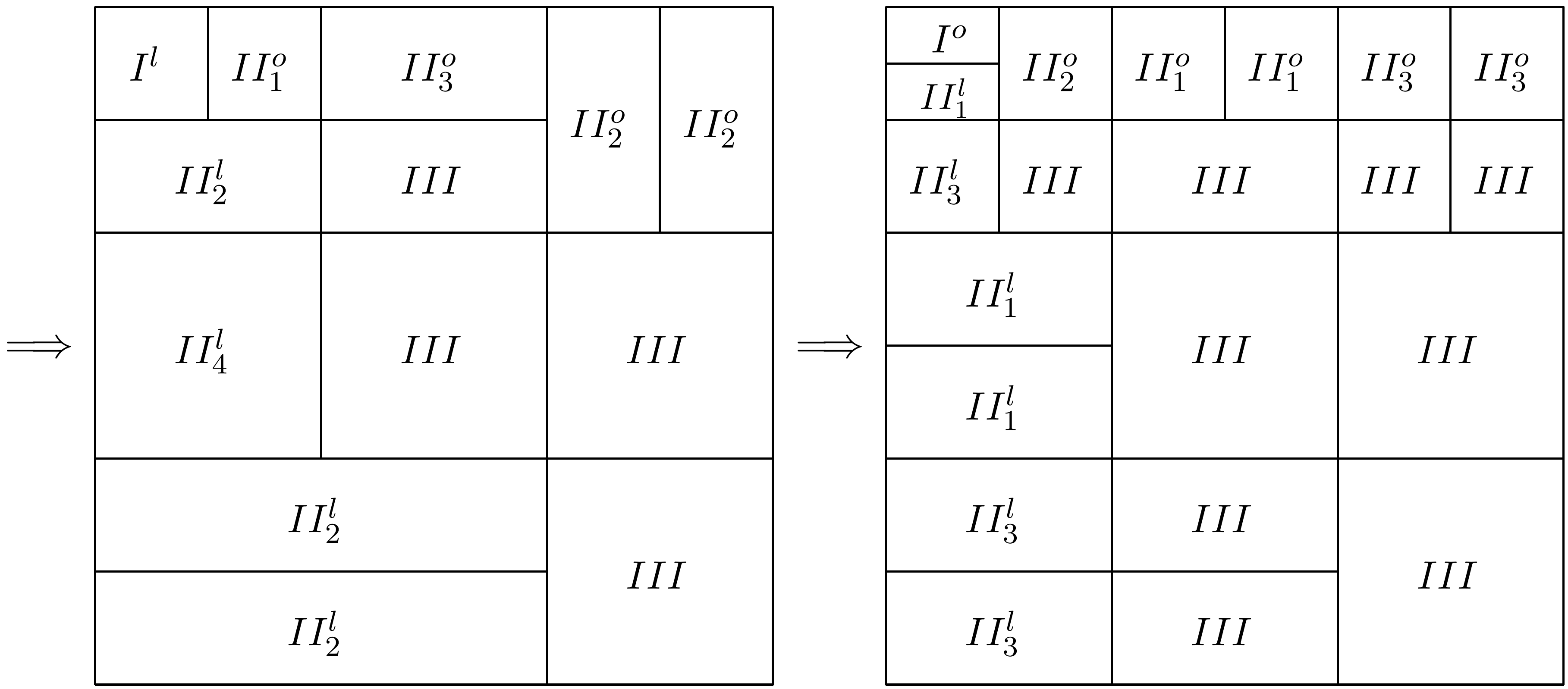
Phascum Cuspidatum II



Phascum Cuspidatum III



Phascum Cuspidatum IV



0L Systems – Definitions I

Definition:
A *Lindenmayer system without interaction* (0L system, for short) is a triple $G = (V, P, \omega)$ where

- V is an alphabet,
- P is a finite complete set of productions over V,
  i.e., P is a finite subset of $V^+ \times V^*$ and,
  for any $a \in V$, there is a word w_a such that $(a, w_a) \in P$,
- $\omega \in V^+$.

instead of (a, w) in P we write $a \to w$

0L Systems – Definitions II

Definition:
Let $G = (V, P, \omega)$ be a 0L system.
For two words $x \in V^+$ and $y \in V^*$, we say that x *directly derives* y in G (written as $x \Longrightarrow_G y$, or $x \Longrightarrow y$ if G is clear from the context) if and only if the following conditions are satisfied:

- $x = x_1 x_2 \dots x_n$ where $x_i \in V$ for $1 \leq i \leq n$,
- $y = y_1 y_2 \dots y_n$,
- $x_i \rightarrow y_i \in P$ for $1 \leq i \leq n$.

0L Systems – Definitions III

$\Longrightarrow^*$ denotes the reflexive and transitive closure of $\Longrightarrow$

Definition:
Let $G = (V, P, \omega)$ be a 0L system.
The language $L(G)$ generated by G is defined as

$$L(G) = \{z \mid \omega \Longrightarrow^* z\}\,.$$

$L_0(G) = \{\omega\}$,
$L_n(G) = \{z \mid v \Longrightarrow z$ for some $v \in L_{n-1}(G)\}$, $n \geq 1$
$L(G) = \bigcup_{n \geq 0} L_n(G)$,

0L Systems — Examples

$G_1 = (\{a\}, \{a \to a^2\}, a)$
$L(G_1) = \{a^{2^n} \mid n \geq 0\}$,

$G_2 = (\{a, b\}, \{a \to \lambda, b \to ab\}, aab)$
$L(G_2) = \{aab, ab\}$,

$G_3 = (\{a\}, \{a \to a, a \to a^2\}, a)$
$L(G_3) = \{a^n \mid n \geq 1\}$,

$G_4 = (\{a, b, c, d, e\}, \{a \to a, b \to ba, c \to cbb, d \to da, e \to cbbd\}, e)$
$L(G_4) = \{e\} \cup \{cbbbababa^2ba^2 \dots ba^n ba^n da^n \mid n \geq 0\}$,

$G_5 = (\{a, b, c\}, \{a \to a^2, b \to ab, c \to bc, c \to c\}, abc)$
$L(G_5) = \{a^{2^n-1}ba^{2^{n_1}-1}ba^{2^{n_2}-1}b \dots a^{2^{n_r}-1}bx \mid$
$n > n_1 > n_2 > \dots n_r > 1, r > 0, x \in \{c, bc\}\}$

Special 0L Systems

Definition:

A 0L system $G = (V, P, \omega)$ is called *propagating* (P0L system, for short) if $a \rightarrow w \in P$ implies $w \neq \lambda$.

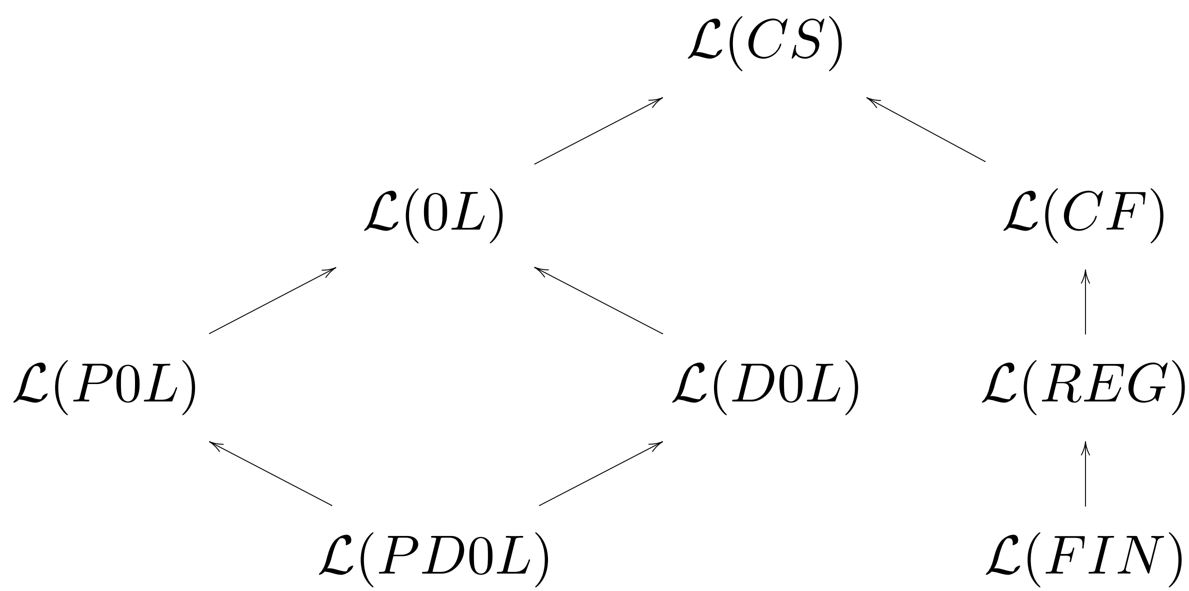
A 0L system $G = (V, P, \omega)$ is called *deterministic* (D0L system, for short) if, for any $a \in V$, $a \rightarrow w \in P$ and $a \rightarrow v \in P$ imply $w = v$.

A PD0L system is a 0L system which is propagating as well as deterministic.

Hierarchy of L Systems Without Interaction

$X \in \{0L,\ P0L, D0L, PD0L\}$,
$\mathcal{L}(X)$ — family of languages generated by X systems

Theorem: The following diagram holds.



Workspace of 0L Systems

Lemma:

Let $G = (V, P, \omega)$ be a 0L system. Then there exists a constant C_G such that, for any word $x \in L(G)$, there is a derivation

$$\omega = w_0 \Longrightarrow w_1 \Longrightarrow w_2 \Longrightarrow \ldots \Longrightarrow w_r = x$$

with $|w_i| \leq C_G \cdot |x|$.

Adult Languages of 0L Systems

Definition: The adult language $L_A(G)$ of a 0L system $G = (V, P, \omega)$ is the set of all words $z \in L(G)$ such that $z \Longrightarrow v$ implies $z = v$.

$G_6 = (\{a, b, c, d, e\}, \{a \to dabc, a \to e, b \to bc, c \to \lambda, d \to e, e \to e\}, a)$

$L(G_6) = \{a, e\} \cup \{e^{n-1}da(bc)^n \mid n \geq 1\} \cup \{e^n(bc)^n \mid n \geq 1\}$

$L_A(G_6) = \{e\} \cup \{e^n(bc)^n \mid n \geq 1\}$

$X \in \{0L, P0L, D0L, PD0L\}$
$\mathcal{L}(AX)$ – family of adult languages generated by X systems

Theorem:
i) $\mathcal{L}(A0L) = \mathcal{L}(AP0L) = \mathcal{L}(CF)$.
ii) $\mathcal{L}(AD0L) = \mathcal{L}(APD0L) = \{\{w\} \mid w \in V^+\} \cup \{\emptyset\}$.

Decision Problems I

Membership problem – given X system $G = (V, P, \omega)$ and $w \in V^*$,
decide whether or not $w \in L(G)$

Emptiness problem – given X system $G = (V, P, \omega)$,
decide whether or not $L(G)$ is empty

Finiteness problem – given X system $G = (V, P, \omega)$,
decide whether or not $L(G)$ is finite

Equivalence problem – given X systems $G = (V, P, \omega)$ and $H = (V, P', \omega')$,
decide whether or not $L(G) = L(H)$

Decision Problems II

Remark: The emptiness problem is not of interest (is trivial) since any 0L system generates a non-empty language.

Theorem:
For 0L systems, the membership problem is decidable in polynomial time.

Theorem:
For 0L systems, the finiteness problem is decidable in polynomial time.

Theorem:
i) For (P)0L systems, the equivalence problem is undecidable.
ii) For (P)D0L systems, the equivalence problem is decidable.

Growth Functions I

For a D0L system G, $L_m(G)$ contains exactly one element w_m.

Definition: The growth function $f_G : \mathbf{N}_0 \to \mathbf{N}_0$ of a deterministic 0L system G is defined by

$$f_G(m) = |w_m| \,.$$

$G = (\{a_1, a_2, \ldots, a_n\}, \{a_1 \to v_1, a_2 \to v_2, \ldots, a_n \to v_n\}, \omega)$
growth matrix of G — $M_G = (\#_{a_i}(v_j))$ (which is an (n,n)-matrix)

Theorem: For a D0L system with growth matrix M_G,

$$f_G(m) = \Psi_V(\omega)(M_G)^m(1,1,\ldots,1)^T \,.$$

Some Linear Algebra

The *characteristic polynomial* $\chi_A(x)$ of a (square) matrix A is defined by

$$\chi(x) = det(A - xE) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

where E is the identity matrix.

We have $a_n = (-1)^n$ and $a_0 = det(A)$

μ is called *eigenvalue* of the square matrix A iff $det(A - \mu E) = 0$.
μ is eigenvalue of the square matrix A iff μ is a root of χ_A.

Cayley-Hamilton Theorem: $\chi_A(A) = 0$

Linear Difference Equations

Let the polynomial $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$ have the roots α_i of multiplicity t_i, $1 \leq i \leq s$.
$(\sum_{i=1}^{s} t_i = n)$

Theorem: The linear difference equation

$$a_nf(m+n)+a_{n-1}f(m+n-1)+\ldots+a_2f(m+2)+a_1f(m+1)x+a_0f(m) = 0$$

with $m \geq 0$ has the solution

$$f(m) = \sum_{i=1}^{s}(\beta_{i,0} + \beta_{i,1}m + \beta_{i,2}m^2 + \ldots \beta_{i,t_i}m^{t_i})\alpha_i^m$$

where $\beta_{i,j}$, $1 \leq i \leq s$, $0 \leq j \leq t_i$ are certain constants.

Growth Functions II

Theorem:

Let $G = (V, P, \omega)$ be a D0L system with $\#(V) = n$.
Let M_G be the growth matrix of G.
For $1 \leq i \leq s$, let μ_i be a eigenvalue of M_G of multiplicity t_i such that $\sum_{i=1}^{s} t_i = n$.
Then

$$f_G(m) = \sum_{i=1}^{s} (\beta_{i,0} + \beta_{i,1} m + \beta_{i,2} m^2 + \ldots \beta_{i,t_i} m^{t_i}) \mu_i^m$$

for certain constants $\beta_{i,j}$, $1 \leq i \leq s$, $0 \leq j \leq t_i$.

Growth Functions III

Theorem:

Let G be a D0L system.
Then the growth f_G satisfies one of the following conditions:

a) there is a constant c such that $f_G(m) \leq c$ for all m,

b) there are constants c_1, c_2 and p such that $c_1 m^p \leq f_G(m) \leq c_2 m^p$ for all m,

c) there are constants c_1 and c_2 such that $c_1^m \leq f_G(m) \leq c_2^m$ for all m.

Lindenmayer Systems with Interaction – Definitions I

Definition: Let k and l be two non-negative integers.
A $< k, l >$ *Lindenmayer system* ($< k, l >$L system for short) is a quadruple $G = (V, \$, P, \omega)$ where

– V is an alphabet, and \$ is a symbol not occurring in V,

– P is a finite set of triples (u, a, v, w) where
 a) $u = \$^r u'$ for some $r \in \mathbf{N}_0$ and some $u' \in V^*$ with $|u'| = k - r$,
 b) $a \in V$,
 c) $v = v'\$^s$ for some $s \in \mathbf{N}_0$ and some $v' \in V^*$ with $|v'| = l - s$,
 d) $w \in V^*$

 and, for any triple (u, a, v) with the properties a), b) and c), there is a $w \in V^*$ such that $(u, a, v, w) \in P$.

– ω is a non-empty word over V.

Lindenmayer Systems with Interaction – Definitions II

Definition: Let G be a $< k, l >$L system.

Let $x \in V^+$ and $y \in V^*$. We say that x *directly derives* y (written as $x \Longrightarrow_G y$) if the following conditions are satisfied:

– $x = a_1 a_2 \ldots a_n$ with $a_i \in V$ for $1 \leq i \leq n$,

– $y = y_1 y_2 \ldots y_n$,

– $(u_i, a_i, v_i) \to y_i \in P$ where

$$u_i = \begin{cases} \$^{k-i+1} a_1 a_2 \ldots a_{i-1} & \text{for } 1 \leq i \leq k \\ a_{i-k} a_{i-k+1} \ldots a_{i-1} & \text{for } k < i \end{cases}$$

$$v_i = \begin{cases} a_{i+1} a_{i+2} \ldots a_{i+l} & \text{for } i + l \leq n \\ a_{i+1} a_{i+2} \ldots a_n \$^{l+i-n} & \text{for } n < i + l \end{cases}$$

The *language* $L(G)$ *generated by* G is defined as $L(G) = \{z \mid \omega \Longrightarrow_G^* z\}$.

Lindenmayer Systems with Interaction – Examples I

$<1,0>$L system $G_7 = (\{a,b,c\},\$,P_7,c)$ with

$$P_7 = \{(\$,a)\to a^2,(\$,b)\to b,(\$,c)\to a,(\$,c)\to ba^2,(a,a)\to a^2\} \cup\{(p,q)\to q \mid (p,q)\in\{a,b,c\}\times\{a,b,c\}\setminus\{(a,a)\}\}$$

$L(G_7)=\{c\}\cup\{a^{2^n}\mid n\geq 0\}\cup\{ba^{2^{n+1}}\mid n\geq 0\}$

$<1,1>$L system $G_8 = (\{a,b\},\$,P_8,ab^2)$ with

P_8 consisting of the following rules: $\begin{cases}(u,a,b)\to a^2 & \text{for } u\in\{a,b,\$\},\\ (a,b,v)\to b^3 & \text{for } v\in\{a,b,\$\},\\ (u,z,v)\to z & \text{in all other cases}\end{cases}$

$L(G_8)=\{a^nb^{2n}\mid n\geq 1\}$

Lindenmayer Systems with Interaction – Examples II

$< 1, 0 >$L system $G_9 = (\{a, b, o, r\}, \$, P_9, ar)$

P_9 consisting of the following rules:

– $(\$, a) \rightarrow o,\ (o, a) \rightarrow b,\ (o, b) \rightarrow o,\ (o, r) \rightarrow ar$,

– $(u, o) \rightarrow a$ for $u \in \{a, b, o, r, \$\}$,

– $(u, z) \rightarrow z$ in all other cases

$$ar \Longrightarrow or \Longrightarrow aar \Longrightarrow oar \Longrightarrow abr \Longrightarrow obr \Longrightarrow aor$$
$$\Longrightarrow oaar \Longrightarrow abar \Longrightarrow obar \Longrightarrow aoar \Longrightarrow oabr \Longrightarrow abbr$$
$$\Longrightarrow obbr \Longrightarrow aobr \Longrightarrow oaor \Longrightarrow aoaar \Longrightarrow oabar \Longrightarrow \ldots$$

Lindenmayer Systems with Interaction – Hierarchies I

$k \in \mathbf{N}_0$ and $l \in \mathbf{N}_0$

$\mathcal{L}(< k, l > L)$ — family of all languages generated by $< k, l >$L systems

$\mathcal{L}(IL) = \bigcup_{k \geq 0, l \geq 0} \mathcal{L}(< k, l > L)$

Corollary:

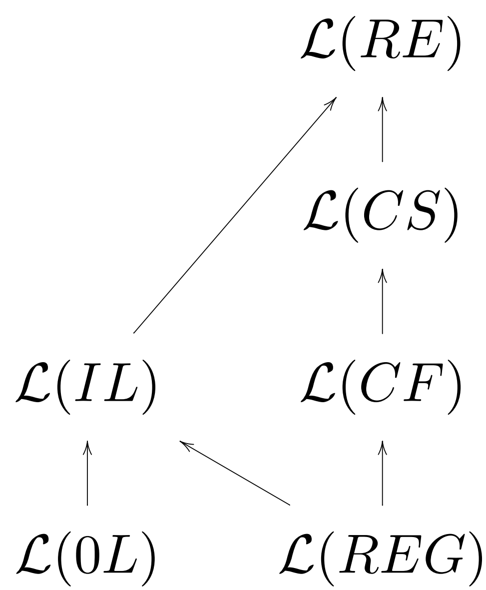
i) $\mathcal{L}(< 0, 0 > L) = \mathcal{L}(0L)$.

ii) For any $k, k', l, l' \in \mathbf{N}_0$, $k \leq k'$ and $l \leq l'$,

$$\mathcal{L}(< k, l > L) \subseteq \mathcal{L}(< k', l' > L) \subseteq \mathcal{L}(IL) \,.$$

Lindenmayer Systems with Interaction – Hierarchies II

Theorem:



Lindenmayer Systems with Interaction – Hierarchies III

Lemma: For any $k, k', l, l' \in \mathbf{N}$ with $k + l = k' + l'$,

$$\mathcal{L}(< k, l > L) = \mathcal{L}(< k', l' > L).$$

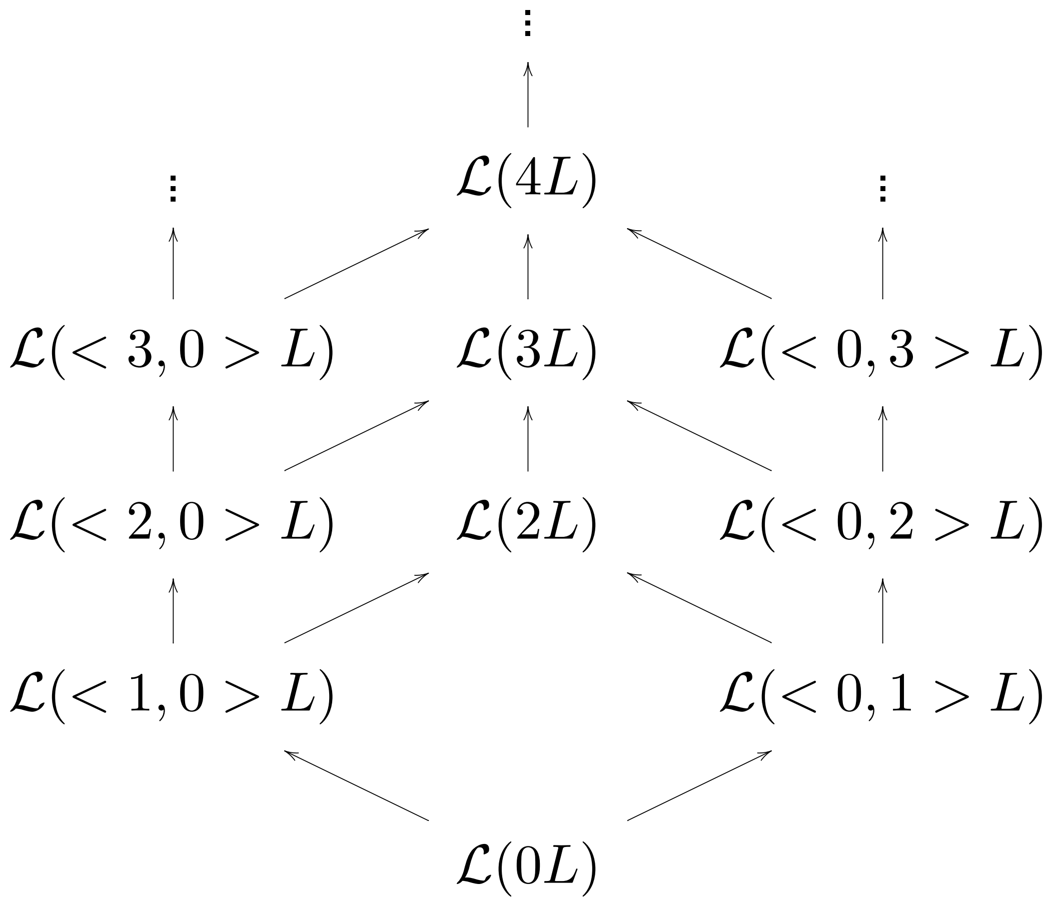
Lemma: For any $k, k', l, l' \in \mathbf{N}_0$ with $k + l < k' + l'$,

$$\mathcal{L}(< k, l > L) \subset \mathcal{L}(< k', l' > L).$$

For $k \geq 2$, $\mathcal{L}(kL) = \mathcal{L}(< 1, k-1 > L)$.

Lindenmayer Systems with Interaction — Hierarchies IV

Theorem:



Lindenmayer Systems with Interaction
Adult Languages and Growth Functions

Theorem: $\mathcal{L}(AIL) = \mathcal{L}(RE)$.

Theorem:
There is a deterministic $< 1, 0 >$L system G such that its growth function is not a growth function of a D0L system.
More precisely, f_G is not bounded by a constant and, for any polynomial p,

$$\lim_{m \to \infty} \frac{f_G(m)}{p(m)} = 0\,.$$