Matrices and Quasi-Matrices

A matrix $(a_{i,j})_{k,l}$ over V is a rectangular scheme with k rows and l columns and the element $a_{i,j} \in V$ is in the meet of the i-th row and the j-th column, $1 \leq i \leq k$ and $1 \leq j \leq l$.

A quasi-matrix $(a_{i,j})_{k_1,k_2,\ldots,k_l}$ over V is a scheme with l columns of length $k_1, k_2, \ldots, k_l$ and the element $a_{i,j} \in V$ is in i-th element of the j-th column (where we count from above to below), $1 \leq i \leq k_j$ and $1 \leq j \leq l$.

Facts:

i) Each matrix $(a_{i,j})_{k,l}$ is a quasi-matrix $(a_{i,j})_{l,l,\ldots,l}$.

ii) Each quasi-matrix $(a_{i,j})_k$ is a matrix $(a_{i,j})_{k,1}$.

Pictorization of Matrices and Quasi-Matrices

Let T be an alphabet.
For two natural numbers $s \geq 1$ and $t \geq 1$, let $CCP_{s,t}$ be the set of all generalized basic chain code pictures p such that, for $(m,n) \in V(p)$, $0 \leq m \leq s$ and $0 \leq n \leq t$.

Let $pic_{s,t} : T \to CCP_{s,t}$ be a mapping.
For a picture p and two integers m und n, let $sh_{m,n}(p)$ be the picture such that $((u,v), b((u,v)) \in p$ iff $((u+m, v+n), b((u+m, v+n)) \in sh_{m,n}(p)$ ($sh_{m,n}(p)$ is obtained by a shift of p by (m,n))
For a quasi-matrix $M = (a_{i,j})_{k_1,k_2,\ldots,k_l}$, we define the picture $Pic(M)$ as the set

$$Pic(M) = \bigcup_{\substack{1 \leq j \leq l \\ 1 \leq i \leq k_j}} sh_{(i-1)s,-jt}\ pic_{s,t}(a_{i,j}).$$

Siromoney Matrix Grammars – Definition I

Definition:
i) A Siromoney matrix grammar is a construct

$$G = (N_1, N_2, I, T, P_1, P_2, S_1, s, t, pic_{s,t})$$

where
– $G_1 = (N_1, I, P_1, S_1)$ is a phrase structure grammar,
– $I \subseteq N_2$,
– for any $i \in I$, $G_i = (N_2, T, P_2, i)$ is a regular grammar in normal form,
– $s, t \in \mathbf{N}$,
– $pic_{s,t} : T \to CCP_{s,t}$.

ii) A Siromoney matrix grammar G is called an X Siromoney matrix grammar if G_1 is an X grammar.

Siromoney Matrix Grammars – Definition II

iii) $M(G)$ is the set of all matrices $(a_{ij})_{k,l}$, $1 \leq i \leq k$, $1 \leq j \leq l$, $k \geq 1$, $l \geq 1$ such that $a_{1j}a_{2j}\ldots a_{kj} \in L(G_{a_j})$ for some $a_1a_2\ldots a_l \in L(G_1)$.

$QM(G)$ is the set of all quasi-matrices $(a_{ij})_{k_1,k_2,\ldots,k_l}$, $l \geq 1$, $k_u \geq 1$ for $1 \leq u \leq l$ such that $a_{1j}a_{2j}\ldots a_{kj} \in L(G_{a_j})$ for some $a_1a_2\ldots a_l \in L(G_1)$.

$PM(G) = \{Pic(M) : M \in M(G)\}$

$PQM(G) = \{Pic(M) : M \in QM(G)\}$

iv) $\mathcal{M}(X)$, $\mathcal{QM}(X)$, $\mathcal{PM}(X)$ and $\mathcal{PQM}(X)$ denote the families of all languages $M(G)$, $QM(G)$, $PM(G)$ and $PQM(G)$, respectively, where G is an X Siromoney matrix grammar.

Relations between Picture Language Families

Theorem:

i) $\mathcal{M}(REG) \subset \mathcal{M}(CF) \subset \mathcal{M}(CS) \subset \mathcal{M}(RE)$,

ii) $\mathcal{QM}(REG) \subset \mathcal{QM}(CF) \subset \mathcal{QM}(CS) \subset \mathcal{QM}(RE)$,

iii) $\mathcal{PM}(REG) \subset \mathcal{PM}(CF) \subset \mathcal{PM}(CS) \subset \mathcal{PM}(RE)$,

iv) $\mathcal{PQM}(REG) \subset \mathcal{PQM}(CF) \subset \mathcal{PQM}(CS) \subset \mathcal{PQM}(RE)$.

Theorem:

i) $\mathcal{PQM}(CF) \subseteq \mathcal{CCP}_{\updownarrow}(CF)$.

ii) $\mathcal{PM}(REG)$ is not contained in $\mathcal{CCP}_{\updownarrow}(CF)$.

iii) $\mathcal{CCP}(REG)$ is not contained in $\mathcal{PM}(CF)$.

"Classical" Decision Problems

Matrix version of the membership problem:
given a matrix M and a Siromoney matrix grammar G, decide whether or not $M \in M(G)$,

Matrix version of the emptiness problem:
given a Siromoney matrix grammar G, decide whether or not $M(G)$ is empty,

Matrix version of the finiteness Problem:
given a Siromoney matrix grammar G, decide whether or not $M(G)$ is finite,

Picture version of the membership problem:
given a picture p and a Siromoney matrix grammar G, decide whether or not $p \in PM(G)$,

Picture version of the emptiness problem:
given a Siromoney matrix grammar G, decide whether or not $PM(G)$ is empty,

Picture version of the finiteness Problem:
given a Siromoney matrix grammar G, decide whether or not $PM(G)$ is finite.

"Classical" Decision Results I

Theorem:
i) The matrix version of the membership problem is decidable for monotone Siromoney matrix grammars.
ii) The matrix version of the membership problem is undecidable for arbitrary Siromoney matrix grammars.

Corollary:
The matrix version of the membership problem for context-free Siromoney matrix grammars is decidable in polynomial time.

Theorem:
i) The picture version of the membership problem is decidable for monotone Siromoney matrix grammars.
ii) The picture version of the membership problem is undecidable for arbitrary Siromoney matrix grammars.

"Classical" Decision Results II

Theorem:
The picture version of the membership problem for regular Siromoney matrix grammars is **NP**-complete.

Theorem:
The matrix and picture versions of the emptiness problem are decidable for context-free Siromoney matrix grammars, and they are undecidable for monotone Siromoney matrix grammars.

Theorem:
The matrix and picture versions of the finiteness problem are decidable for context-free Siromoney matrix grammars, and they are undecidable for monotone Siromoney matrix grammars.

Submatrix and Subpicture Problem

Submatrix Problem:
Given: Siromoney matrix grammar G, matrix M
Question: Is there a matrix $M' \in M(G)$ such that M is a submatrix of M'

Subpicture Problem:
Given: Siromoney matrix grammar G, chain code picture p
Question: Is there a matrix $M' \in M(G)$ such that p is a subpicture of $Pic(M')$

Theorem: For context-free Siromoney matrix grammars and arbitrary matrices, the submatrix problem is decidable in polynomial time.

Theorem: For context-free Siromoney matrix grammars and arbitrary pictures, the subpicture problem is decidable.

Theorem: The subpicture problem is NP-complete for regular Siromoney matrix languages.

The languages L_M and $L_{\neg M}$

For a matrix language L and a matrix M we set

$$\begin{aligned} L_M &= \{M' \mid M' \in M(G),\ M \text{ is a submatrix of } M'\}, \\ L_{\neg M} &= \{M' \mid M' \in M(G),\ M \text{ is not a submatrix of } M'\}. \end{aligned}$$

Lemma: There are a language $L \in \mathcal{M}(REG)$ and matrices M and M' such that $L_M \notin \mathcal{M}(CF)$ and $L_{\neg M'} \notin \mathcal{M}(CF)$.

Lemma: For $X \in \{REG, CF\}$, any Siromoney matrix language $L \in \mathcal{M}(X)$ and any $(m,1)$-matrix M, the languages L_M and $L_{\neg M}$ are in $\mathcal{M}(X)$.

Lemma: For $X \in \{REG, CF\}$, any Siromoney matrix grammar G of type X such that $L(G_A)$ is finite for any $A \in I$ and any matrix M, the languages $L(G)_M$ and $L(G)_{\neg M}$ are in $\mathcal{M}(X)$.

Universal Submatrix Problem

Universal Submatrix Problem:
Given: Siromoney matrix grammar G, matrix M
Question: Is M a submatrix of any $M' \in M(G)$

Theorem: For context-free Siromoney matrix grammars and arbitrary $(m, 1)$-matrices, the universal submatrix problem is decidable.

Theorem: For context-free Siromoney matrix grammars such that $L(G_A)$ is finite for any $A \in I$ and arbitrary matrices, the universal submatrix problem is decidable.

Theorem: For regular Siromoney matrix grammars and arbitrary matrices (with at most two columns), the universal submatrix problem is undecidable.

Universal Subpicture Problem

Universal Subpicture Problem:
Given: Siromoney matrix grammar G, picture p
Question: Is p a subpicture of $Pic(M')$ for any $M' \in M(G)$

Theorem: For regular Siromoney matrix grammars and any matrix (with at most two columns), the universal subpicture problem is undecidable.

Decidability of "geometric" properties I

Theorem:

It is undecidable for regular Siromoney grammars whether or not $PM(G)$ contains

i) a connected picture,

ii) a 2-regular picture,

iii) a Eulerian picture,

iv) a Hamiltonian picture,

v) a tree.

Decidability of "geometric" properties II

Theorem: It is decidable for regular Siromoney grammars whether or not all picture of $PM(G)$ are

i) k-regular pictures for $k \in \{1,2\}$,

ii) edge colourable by k colours for $k \in \{1,2,3\}$.

(We say that a chain code picture p is edge-colourable by k colours, if there is a mapping from the set of unit lines of p to $\{1,2\ldots,k\}$ such that any two different unit lines of p which intersect in a node are mapped to different numbers.)

Theorem: It is undecidable for regular Siromoney grammars whether or not all picture of $PM(G)$ are

i) connected pictures,

ii) Eulerian pictures,

iii) Hamiltonian pictures.